MATHEMATICS NOTES

Note 44

January 1977

Approximations for Terms Related to the Kernel in Thin-Wire Integral Equations

A. J. Poggio
R. W. Adams

Lawrence Livermore Laboratory
Livermore, California

ABSTRACT

Difficulties can arise in the evaluation of elements in the impedance matrix associated with a moment-method solution of the thin-wire electric field integral equation. The results of a detailed study are presented and several representations are considered and compared. From information regarding errors, regions of acceptability are obtained The deficiencies in the thin-wire kernel are clearly illustrated and remedies are suggested.

ACKNOWLEDGMENT

The Air Force Weapons Laboratory supported this study under Project Order 74-212.

Contents

Illustrations

Illustrations (Continued)

Tables

SECTION I
Introduction

The kernel used in integral equations can often represent the analyst's greatest difficulty. In electromagnetics, it contains a singularity over which an integration must be performed, and as a result, extreme care must be exercised in performing the integration.

In this report, the defining integral equation in terms of electric field for problems involving conductors is

$$\hat{n} \times \bar{E}^{inc}(\bar{r}) = - \frac{1}{4\pi j\omega\varepsilon} \hat{n} \times \int_s \bar{J}(r') \cdot (\nabla'\nabla' + k^2\bar{\bar{I}}) g(r,\bar{r}') \, d^2r' \; (\bar{r}\varepsilon s). \quad (1)$$

Thus, for a cylinder of radius a with azimuthally symmetric excitation that is axially aligned ($\hat{z}$), one must have

$$E_z^{inc}(z) = - \frac{1}{4\pi j\omega\varepsilon} \int_s (I(z') \left[\frac{\partial^2}{\partial z'^2} + k^2\right] \frac{1}{2\pi a} g(\bar{r},\bar{r}') \, d^2r' \; (z\varepsilon s), \quad (2)$$

with the azimuthally directed component of current density existing only as the solution of a homogeneous equation. In this expression, $k = 2\pi/\lambda$, i.e., the wavenumber in the ambient medium. Hence, for a circular cylinder,

$$E_z^{inc}(z) = - \frac{1}{j\omega\varepsilon} \int_c dz' \, I(z') \left(\frac{\partial^2}{\partial z'^2} + k^2\right) \frac{1}{8\pi^2} \int_0^{2\pi} d\phi' \, g(z,z' \; \phi,\phi') \; (z\varepsilon c), \quad (3)$$

with

$$g(z,z', \; \phi = 0,\phi') = \frac{e^{-jk\sqrt{(z-z')^2+4a^2\sin^2\phi'/2}}}{\sqrt{(z-z')^2+4a^2\sin^2\phi'/2}} . \quad (4)$$

Equation (3) with Eq. (4) is an exact representation in this case, as there are no approximations involved in the derivation, and, hence, there are no restrictions on any parameters in the problem.

In this report, we define the kernel of Eq. (3) to be

$$G(z,z') = \frac{1}{8\pi^2}\int_0^{2\pi} d\phi' g(z,z',\phi = 0,\phi')\, d\phi', \tag{5}$$

that is, a term proportional to the azimuthal integral of the free-space Green's function. There is a great deal of interest in not only the approximations of the kernel G(z,z') but also in the value of the double integral in Eq. (3). However, the integral

$$\int_{z'} dz'\, I(z') \left(\frac{\partial^2}{\partial z'^2} + k^2\right) G(z,z') \tag{6}$$

has an analytical solution in terms of values at the end of the range of integration in z' when I(z') is of sinusoidal (sin kz' or cos kz') form.[1] Therefore, we initially restrict our attention to the evaluation of the integrals in Eq. (3) for the case where I(z') is a constant this gives rise to nonanalytically integrable integrands. We are interested in the value of Q(z), i.e.,

$$Q(z) = \int_\alpha^\beta dz' \frac{1}{8\pi^2}\int_0^{2\pi} d\phi'\, g(z,z',\phi = 0,\phi'). \tag{7}$$

Because the major difficulties in the evaluation of Q(z) occur when the singularity in g(z,z', ϕ, ϕ') is in the range [α,β], we focus most closely on this range. However, difficulties have also been observed when the singularity is outside, but close to, this range. We comment on this difficulty where appropriate.

Two basic options exist in the evaluation of Q(z). Specifically, they involve the order in which the ϕ' and z' integrations in Eq. (7) are performed. In this report, we will deal with three different approaches to the evaluation, two of which evaluate the integrals in the order indicated in Eq. (7).

Alternatives also exist regarding the manner in which g(z, z', ϕ, ϕ') is written. One of the basic goals of this report is the definition of some forms for Q(z) that satisfy certain accuracy requirements over ranges of the

variables α/λ, β/λ, and the radius of the cylinder normalized to wavelength (a/λ). It is evident, both in the derivations and the numerical computations, that the particular representations used for $g(z, z', \phi, \phi')$ can affect the ultimate accuracy in $Q(z)$.

For purposes of comparison, we derive three distinct approaches that lead to three different forms of $Q(z)$. Naturally, each approach includes several different results, each of which is the outcome of retaining higher-order terms in series representations. Also, because the values of $G(z,z')$ and $\partial G/\partial z$ are needed in certain cases, these functions are considered when appropriate. For certain approximations, the results of numerical computations are analyzed and the attendant errors studied.

SECTION II
Representations for Q(z)

THE CLASSICAL APPROACH

The classical approach has been the most widely used and was one of the first introduced in the study of the kernel of the integral equations with which we are concerned. The essence of the method is described in Schelkunoff and Friis,[2] and is explicitly carried out to various orders of approximation in Poggio and Mayes.[3] In the latter work, $g(z, z', \phi, \phi')$ as given by Eq. (4), is written in the form

$$g(z,z',\phi,\phi') = \frac{1}{R} - \frac{1 - e^{-jkR}}{R}, \tag{8}$$

with

$$R = \sqrt{4a^2 \sin^2 \frac{\phi'}{2} + (z-z')^2}.$$

Attention is then focussed on the evaluation of the kernel

$$G(z,z') = \frac{1}{8\pi^2} \int_0^{2\pi} d\phi' g(z,z',\phi = 0,\phi'). \tag{9}$$

when the last term in Eq. (8) is expanded in a Maclaurin series about $kR = 0$.

A derivation and comparison of the various approximations to $G(z,z')$ are provided in Appendix A and are given in Table 1.

---

[1] J. A. Stratton, Electromagnetic Theory, (McGraw Hill, New York, 1941).

[2] S. A. Schelkunoff and H. T. Friis, Antenna Theory and Practice, (John Wiley and Sons, Inc., New York, 1952).

[3] A. J. Poggio, and P. E. Mayes, Numerical Solution of Integral Equations of Dipole and Slot Antennas Including Active and Passive Loading, Air Force Avionics Laboratory Rept., Wright-Patterson Air Force Base, Ohio (1969).

Table 1. Approximations to the kernel.

$$G_{a1} = \frac{1}{8\pi^2 a}\left[2k_1 K(k_1) - j\,4\pi^2\, a/\lambda\right] \qquad k_1 = \frac{2a}{\sqrt{4a^2+(z-z')^2}}$$

$$G_{a2} = \frac{1}{8\pi^2 a}\left(2k_1 \ln\frac{8a}{|z-z'|} - j\,4\pi^2\, a/\lambda\right)$$

$$G_{a3} = \frac{1}{8\pi^2 a}\left(2 \ln\frac{8a}{|z-z'|} - j\,4\pi^2\, a/\lambda\right)$$

$$G_b = \frac{1}{8\pi^2 a}\left[2k_1 K(k_1) - j\,4\pi^2\, a/\lambda - \frac{16\pi^2}{k_1}(a/\lambda)^2 E(k_1)\right]$$

$$G_c = \frac{1}{8\pi^2 a}\left[2k_1 K(k_1) - j\,4\pi^2\, a/\lambda - \frac{16\pi^2}{k_1}(a/\lambda)^2 E(k_1) - j\,\frac{16\pi^4}{3}(a/\lambda)^3 + j\,\frac{8}{3}\pi^4 (a/\lambda)^3 \cdot \left(\frac{z-z'}{a}\right)^2\right]$$

$$G_d = \frac{1}{8\pi^2 a}\left[2\pi \frac{e^{-jka\sqrt{1+\left(\frac{z-z'}{a}\right)^2}}}{\sqrt{1+\left(\frac{z-z'}{a}\right)^2}}\right]$$

$$G_e = \frac{1}{8\pi^2 a}\left(2\pi \frac{e^{-jka\,\left|\frac{z-z'}{a}\right|}}{\left|\frac{z-z'}{a}\right|}\right)$$

$$G = \frac{1}{8\pi^2 a}\left[\int_0^{2\pi} \frac{e^{-jka\sqrt{4\sin^2\frac{\phi'}{2}+\left(\frac{z-z'}{a}\right)^2}}}{\sqrt{4\sin^2\frac{\phi'}{2}+\left(\frac{z-z'}{a}\right)^2}}\, d\phi'\right]$$

Using any of these approximations, one is still faced with the z' integral indicated in Eq. (7). The first three representations, G_{a1}, G_{a2}, and G_{a3}, are independent of a/λ and the first term in the approximation of $G(z,z')$ for $z \approx z'$ represents a static interaction. These three approximations derive from the decomposition of $\phi(z,z', \phi = 0,\phi')$, in the form given in Eq. (8), with the retention of only the first term of the MacLaurin series in the $(1 - e^{-jkR})/R$ term. The $1/R$ term yields an exact form, namely, an elliptic integral, while the second term is $O(1/\lambda)$. The representations are written in an ascending series in $1/\lambda^n$.

AN ALTERNATIVE APPROACH

The foundation for the derivation of an alternative representation of $Q(z)$ for small $(z-z')$ has been provided by Tesche.[4] The approach consists of first evaluating the z' integral in Eq. (7) and then performing the ϕ' integration.

In this representation,

$$Q(z) = \frac{1}{8\pi^2} \int_0^{2\pi} d\phi' \, P(z,\phi'), \tag{10}$$

where

$$P(z,\phi') = \int_\alpha^\beta dz' \, g(z,z',\phi = 0,\phi'). \tag{11}$$

The derivation, detailed in Appendix B, requires an expansion of $g(z,z', \phi, \phi')$ about $kR = 0$ and yields, for $P(z,\phi')$,

$$P(z,\phi') = - \sum_{m=0}^{\infty} \frac{(-jk)^m}{m!} \int_{z-\alpha}^{z-\beta} d\xi \left(4a^2 \sin^2 \frac{\phi'}{2} + \xi^2\right)^{\frac{m-1}{2}} \tag{12}$$

The case where the singularity is at the center of the range in ξ is of particular interest because, in a collocation solution of the integral equation, the observation points are usually placed at the center of the

[4] F. M. Tesche, Evaluation of the Surface Integral Occurring in the E-field Integral Equations for Thin-Wire Antennas, Air Force Weapons Laboratory, Kirtland Air Force Base, New Mexico, Mathematics Note 29 (1973).

intervals. For this case, $\alpha = z - \Delta/2$ and $\beta = z + \Delta/2$, and the results for $Q(z)$ are somewhat simplified. We include below a tabulation of the representations of $Q(z)$, denoting them $Q_m(z)$ to indicate the order of the terms retained in the derivation. Details are in Appendix B.

$$Q_1(z) = \frac{-jk\Delta}{4\pi} + \frac{1}{2\pi}\,\ell n\ 2 + \frac{1}{\pi^2}\int_0^{\pi/2} d\psi\ \ell n\left[\frac{\Delta}{4a} + \sqrt{\sin^2\psi + \left(\frac{\Delta}{4a}\right)^2}\right] . \qquad (13)$$

$$Q_2(z) = \frac{-jk\Delta}{4\pi} + \frac{\ell n2}{2\pi}\left[1 - \frac{(ka)^2}{2}\right] + \frac{1}{8\pi}(ka)^2 - \frac{1}{4\pi^2}(k\Delta)(ka)\sqrt{1 + \left(\frac{\Delta}{4a}\right)^2}$$
$$E\left[\frac{1}{\sqrt{1 + \left(\frac{\Delta}{4a}\right)^2}}\right] + \frac{1}{\pi^2}\int_0^{\pi/2} d\psi(1 - k^2a^2\sin^2\psi)\ \ell n\left[\frac{\Delta}{4a} + \sqrt{\sin^2\psi + \left(\frac{\Delta}{4a}\right)^2}\right] . \qquad (14)$$

$$Q_3(z) = \frac{-jk\Delta}{4\pi}\left(1 - \frac{k^2a^2}{3} - \frac{k^2\Delta^2}{72}\right) + \frac{\ell n2}{2\pi}\left(1 - \frac{k^2a^2}{2}\right)$$
$$+ \frac{1}{8\pi}(ka)^2 - \frac{1}{4\pi^2}(k\Delta)(ka)\sqrt{1 + \left(\frac{\Delta}{4a}\right)^2}\ E\left(\frac{1}{\sqrt{1 + \left(\frac{\Delta}{4a}\right)^2}}\right)$$
$$+ \frac{1}{\pi^2}\int_0^{\pi/2} d\psi(1 - k^2a^2\sin^2\psi)\ \ell n\left[\frac{\Delta}{4a} + \sqrt{\sin^2\psi + \left(\frac{\Delta}{4a}\right)^2}\right] . \qquad (15)$$

Note that $Q_1(z)$ is independent of ka but dependent on $k\Delta$ and the ratio Δ/a; that $Q_2(z)$ has terms of order ka and $(ka)^2$; and that the integrand in $Q_3(z)$ is only slightly more complicated than that in $Q_1(z)$. Furthermore, all the integrands are nonsingular and can be evaluated numerically with ease.

AN EXTENDED, THIN-WIRE KERNEL

Deficiencies have been observed in the thin-wire kernel, as illustrated in Appendix A. There, the thin-wire kernel given by

$$G(z,z') = \frac{e^{-jk\sqrt{(z-z')^2+a^2}}}{4\pi\sqrt{(z-z')^2+a^2}} \tag{16}$$

is not a suitable representation of the exact $G(z,z')$ of Eqs. (4) and (5) for small values of $|z-z'|/a$.

In our investigation of representations of $G(z,z')$ or $Q(z)$, we rederived and extended the thin-wire kernel. An integral representation of the function $g(z,z',\phi = 0,\phi')$ is available in the form[5]

$$\frac{e^{-jk\sqrt{(z-z')^2+\rho^2}}}{\sqrt{(z-z')^2+\rho^2}} = \frac{1}{2j}\int_{-\infty}^{\infty} d\nu\, e^{j\nu|z-z'|}\, H_o^{(2)}\left(\rho\sqrt{k^2-\upsilon^2}\right), \tag{17}$$

where

$$\rho = 2a \sin\frac{\phi'}{2} .$$

The corresponding integral form for $G(z,z')$, with an interchange in the order of integration, is therefore

$$G(z,z') = \frac{1}{16\pi^2 j}\int_{-\infty}^{\infty} d\nu\, e^{j\nu|z-z'|}\int_0^{2\pi} d\phi'\, H_o^{(2)}\left(2a\sin\frac{\phi'}{2}\sqrt{k^2-\nu^2}\right) . \tag{18}$$

The ϕ' integral has a well-known result so that

$$G(z,z') = \frac{1}{8\pi j}\int_{-\infty}^{\infty} d\nu\, e^{j\nu|z-z'|}\, J_o\left(a\sqrt{k^2-\nu^2}\right)\, H_o^{(2)}\left(a\sqrt{k^2-\nu^2}\right) . \tag{19}$$

---

[5] I. S. Gradshteyn and I. M. Ryzhik, Tables of Integrals, Series, and Products (Academic Press, New York, 1965).

This approach to obtain the exact integral representation of G(z,z') is outlined in Hallen.[6]

The substitution of the series representation for the zeroeth-order Bessel function, i.e.,

$$J_o\left(a\sqrt{k^2-\nu^2}\right) = \sum_{i=0}^{\infty} (-1)^i \frac{a^{2i}}{2^{2i} i!\ \Gamma(i+1)} \left(k^2-\nu^2\right)^i , \tag{20}$$

allows Eq. (19) to be written as

$$G(z,z') = \frac{1}{8\pi j} \sum_{i=0}^{\infty} \frac{(-1)^i a^{2i}}{2^{2i} i!\ \Gamma(i+1)} \int_{-\infty}^{\infty} d\nu\ e^{j\nu|z-z'|} \left(k^2-\nu^2\right)^i H_o^{(2)}\left(\varepsilon\sqrt{k^2-\nu^2}\right) \tag{21}$$

Furthermore, one can introduce the series representation

$$\left(k^2-\nu^2\right)^i = k^{2i} \sum_{\ell=0}^{i} (-1)^\ell \frac{i!}{\ell!(i-\ell)!} \nu^{2\ell} k^{-2\ell} \tag{22}$$

and obtain

$$G(z,z') = \frac{1}{8\pi j} \sum_{i=0}^{\infty} (-1)^i \frac{(ka)^{2i}}{2^{2i} i!\ \Gamma(i+1)} \sum_{\ell=0}^{i} (-1)^\ell \frac{i!}{\ell!(i-\ell)!}$$

$$k^{-2\ell} \int_{-\infty}^{\infty} d\nu \nu^{2\ell}\ e^{j\nu|z-z'|}\ H_o^{(2)}\left(a\sqrt{k^2-\nu^2}\right) . \tag{23}$$

However, the ν integral is nothing more than

$$(-1)^\ell \frac{\partial^{2\ell}}{\partial z^{2\ell}} \int_{-\infty}^{\infty} d\nu\ e^{j\nu|z-z'|}\ H_o^{(2)}\left(a\sqrt{k^2-\nu^2}\right) , \tag{24}$$

---

[6] E. Hallen, Electromagnetic Theory (John Wiley and Sons, 1962).

and, using Eq. (17),

$$G(z,z') = \frac{1}{4\pi} \sum_{i=0}^{\infty} (-1)^i \frac{(ka)^{2i}}{2^{2i} i!\ \Gamma(i+1)} \sum_{\ell=0}^{i} \frac{i!}{\ell!(i-\ell)!} k^{-2\ell} \frac{\partial^{2\ell}}{\partial z^{2\ell}} \frac{e^{-jk\sqrt{(z-z')^2+a^2}}}{\sqrt{(z-z')^2+a^2}} . \tag{25}$$

This expression for $G(z,z')$ can be written in the compact form

$$G(z,z') = \frac{1}{4\pi} \sum_{i=0}^{\infty} (-1)^i \frac{(ka)^{2i}}{2^{2i} i!\ \Gamma(i+1)} \left(1 + \frac{1}{k^2} D^2\right)^i g_{thin}, \tag{26}$$

where

$$g_{thin} = \frac{e^{-jk\sqrt{(z-z')^2+a^2}}}{\sqrt{(z-z')^2+a^2}}$$

and

$$D^2 = \frac{\partial^2}{\partial z^2} .$$

In Eq. (25), the $i = 0$ term corresponds to the common, thin-wire form

$$G_0(z,z') = \frac{1}{4\pi} g_{thin} , \tag{27}$$

while retaining the $i = 0$ and $i = 1$ terms corresponds to the "extended" thin-wire kernel

$$G_1(z,z') = \frac{1}{4\pi} \left[1 - \frac{(ka)^2}{4} \left(1 + \frac{\partial^2}{\partial(kz)^2}\right)\right] g_{thin} . \tag{28}$$

Expanding the terms in $G_1(z,z')$ yields

$$G_1(z,z') = \frac{1}{4\pi}\frac{e^{-jk\sqrt{(z-z')^2+a^2}}}{\sqrt{(z-z')^2+a^2}}\left\langle 1 - \frac{1+jka\sqrt{1+\left(\frac{z-z'}{a}\right)^2}}{2\left[1+\left(\frac{z-z'}{a}\right)^2\right]}\right.$$

$$\left. + \frac{1}{4\left[1+\left(\frac{z-z'}{a}\right)^2\right]^2}\left\{3jka\sqrt{1+\left(\frac{z-z'}{a}\right)^2} + 3 - k^2a^2\left[1+\left(\frac{z-z'}{a}\right)^2\right]\right\}\right\rangle \qquad (29)$$

The extended thin-wire kernel derived above does not exhibit the appropriate singularity at $z = z'$ and, hence, cannot be expected to yield suitably accurate results for $Q(z)$ near that point. We can expect however, that its range of validity will exceed that pertaining to the thin-wire kernel.

THE SMALL DISTANCE LIMIT OF THE THIN-WIRE FORM

The thin-wire form for $G(z,z')$ was derived in the previous section. We now consider an approximation to $Q(z)$ that is obtained by expanding the exponential in a Maclaurin series. This approach is presented by Harrington[7] and used for comparison by Tesche.[4]

We are interested in the approximation

$$Q_4(z) = \int_\alpha^\beta dz' \frac{1}{8\pi^2}\int_0^{2\pi} d\phi' \frac{e^{-jk\sqrt{(z-z')^2+a^2}}}{\sqrt{(z-z')^2+a^2}} . \qquad (30)$$

Using the Maclaurin series, one obtains the integral

$$Q_4(z) = \frac{1}{4\pi}\int_\alpha^\beta dz' \left[\frac{1}{\sqrt{(z-z')^2+a^2}} - jk\right] , \qquad (31)$$

[7] R. F. Harrington, Field Computation by Moment Methods (MacMillan, 1968).

which can be evaluated to yield

$$Q_4(z) = \frac{1}{4\pi}\left[\ell n \frac{z-\alpha + \sqrt{(z-\alpha)^2 + a^2}}{z-\beta + \sqrt{(z-\beta)^2 + a^2}} - jk\,(\beta-\alpha)\right] . \qquad (32)$$

For $\beta = z + \Delta/2$, $\alpha = z - \Delta/2$, and $\Delta >> a$,

$$Q_4(z) = \frac{1}{2\pi} \ell n \frac{\Delta}{a} - \frac{1}{4\pi} jk\Delta . \qquad (33)$$

SECTION III
A Numerical Comparison of Various Kernels

To compare the various kernels or the functions Q(z), numerical computations are necessary. We performed these computations for selected representations and can therefore suggest forms that satisfy certain accuracy constraints.

In the numerical studies, we included the functions Q(z) shown below.

Table 2. The functions and their definitions.

$Q(z)$	Eq. (7) and Eq. (4)(Exact)
$Q_0(z)$	$\int dz'\, G_0(z,z')$ Eq. (27)
$Q_1(z)$	Eq. (13)
$Q_2(z)$	Eq. (14)
$Q_3(z)$	Eq. (15)
$Q_4(z)$	Eq. (32)
$Q_e(z)$	$\int dz'\, G_1(z,z')$ Eq. (28)

The relative error is used as a criterion of accuracy in the study of the representations and is defined as

$$\text{Relative Error} = \left| \frac{Q - Q_i}{Q} \right| , \tag{34}$$

where Q is the exact representation [Eqs. (7) and (4)] and Q_i is the respective approximation. Because the error is dependent on z, α, β, k, and a, we restrict our attention to

- Self terms: $\alpha = z - \Delta/2,\ \beta = z + \Delta/2$
- Adjacent terms: $\alpha = z + \Delta/2,\ \beta = z + 3\Delta/2$
- Next-to-adjacent terms: $\alpha = z + 3\Delta/2,\ \beta = z + 5\Delta/2,$

with variations in wavenumber k and radius a. In this manner, the dependency of Q on z is contained implicitly in the specific term we are considering and in the range $[\alpha,\beta]$. Therefore, we have parameterized our curves with respect to the term considered and to ka, and have plotted the relative error versus Δ/a.

RELATIVE ERRORS IN THE SELF TERMS

The relative errors in the evaluation of the self terms, $Q(z)$ for $\alpha = z - \Delta/2$ and $\beta = z + \Delta/2$, have been calculated. The results are shown in Figs. 1 through 8: the relative errors are plotted vs Δ/a, $k\Delta$, and N/λ. The latter abscissa is the number of intervals contained in one wavelength. Note that $k\Delta = (ka)(\Delta/a)$ and $N/\lambda = 2\pi/k\Delta$. In Figs. 1 through 5 there is a region denoting error bounderies for certain representations. The relative error curve for each representation oscillated within these boundaries and the detailed nature of the curves themselves did not convey any seemingly useful information.

The range of values of Δ/a shown in Figs. 1 and 2 do not in reality correspond to the values likely to be encountered in practice. Because one generally uses on the order of five to tens of intervals per wavelength, most of the representations will suffice. In fact, most will provide relative errors less than 1% in the commonly used range of N/λ.

For thicker wires, the relative accuracies begin to deteriorate. Although the onset is seen in Fig. 3, it is more clearly seen in Figs. 4 through 8 where the relative errors over the indicated range of Δ/a are not monotonic functions and often show a decreasing, and then an increasing, relative error as Δ/a decreases. The deficiency in Q_4 is evident in Figs. 4 through 8. Furthermore, the relative inaccuracy in Q_1, Q_2, and Q_3 is seen to increase with increasing ka, and $k\Delta$ (in general). Inspection of the thin-wire representation also indicates a problem as its relative error curves exhibit minima. However, the relative error in the extended thin-wire representation is reasonably constant with respect to variations in ka and the relative error is less than 1% for $\Delta/a \gtrsim 2$.

A careful study of Figs. 1 through 8 allows us to establish the set of representations that will yield relative errors less than 1% over the ranges $10^{-4} \leq ka \leq 0.4$, and $0.01 \leq k\Delta \leq 1.0$ (Fig. 9).

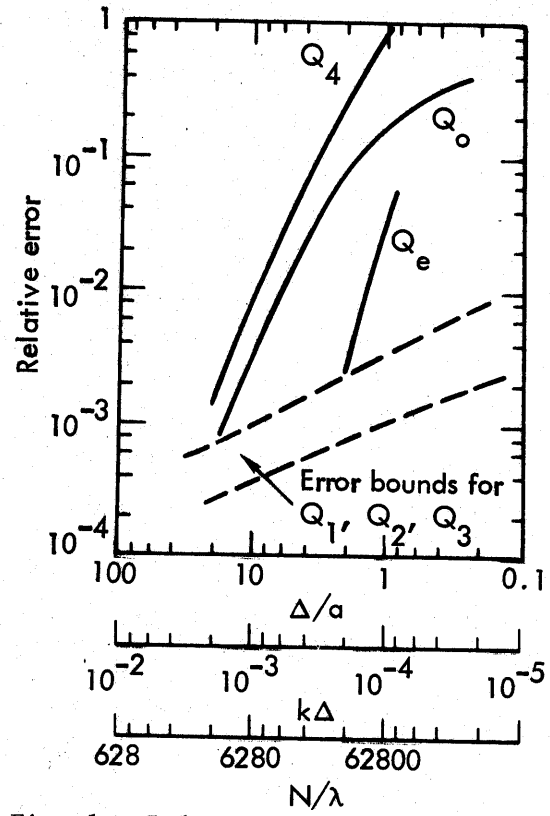

Fig. 1. Relative errors in various Q_i's (self term) for ka = 10^{-4}.

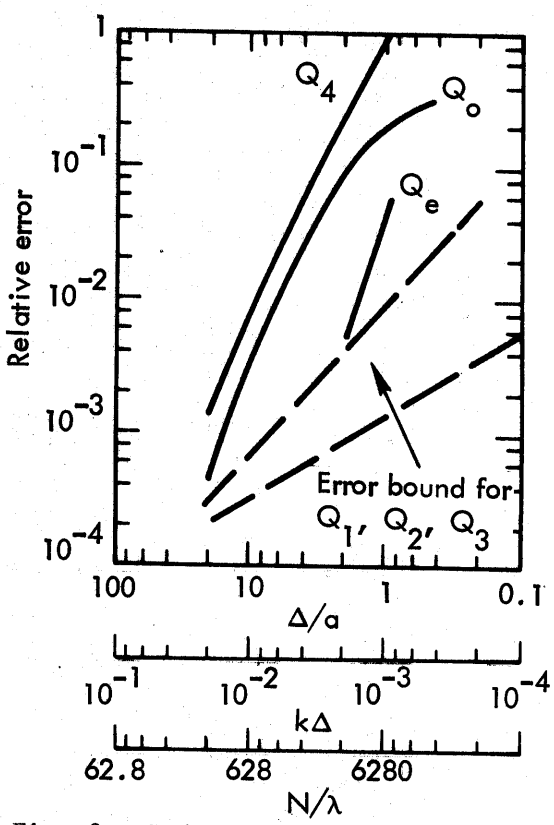

Fig. 2. Relative errors in various Q_i's (self term) for ka = 10^{-3}.

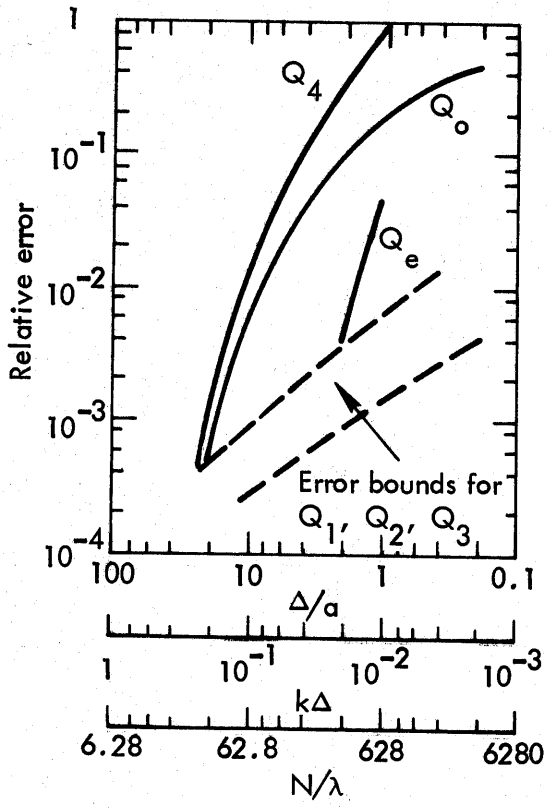

Fig. 3. Relative errors in various Q_i's (self term) for ka = 10^{-2}.

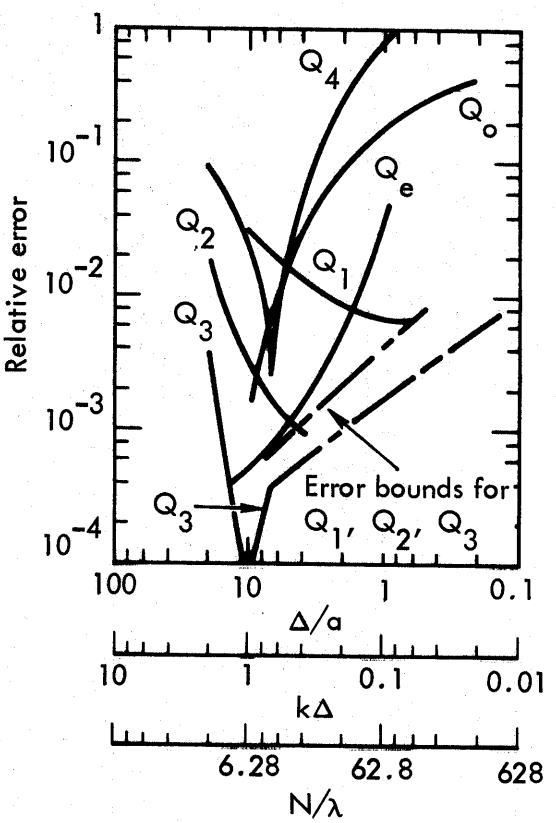

Fig. 4. Relative errors in various Q_i's (self term) for ka = 10^{-1}.

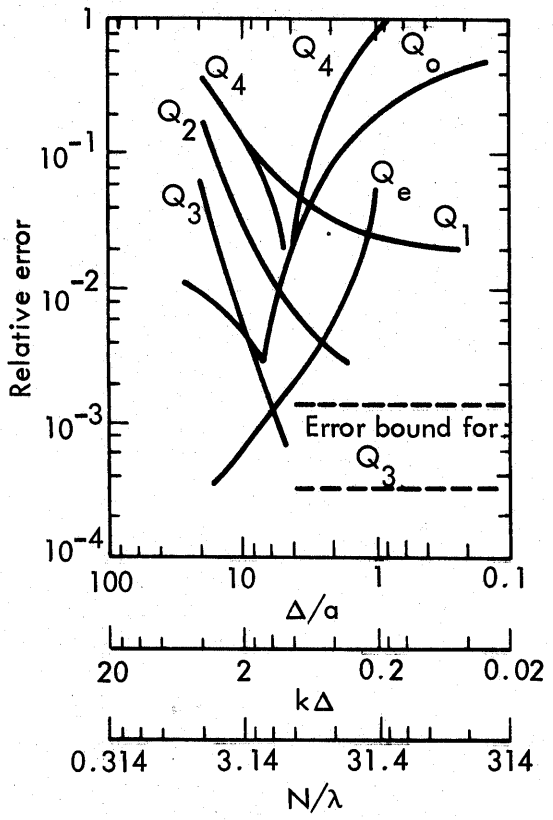

Fig. 5. Relative errors in various Q_i's (self term) for ka = 0.2.

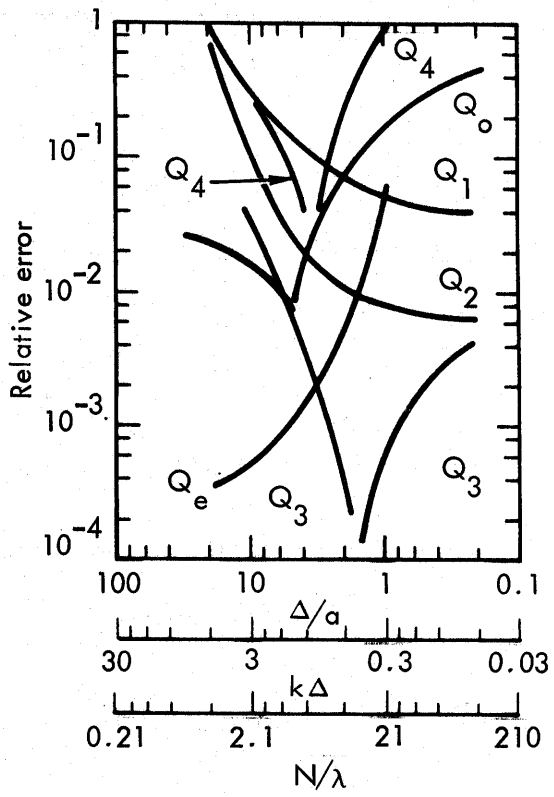

Fig. 6. Relative errors in various Q_i's (self term) for ka = 0.3.

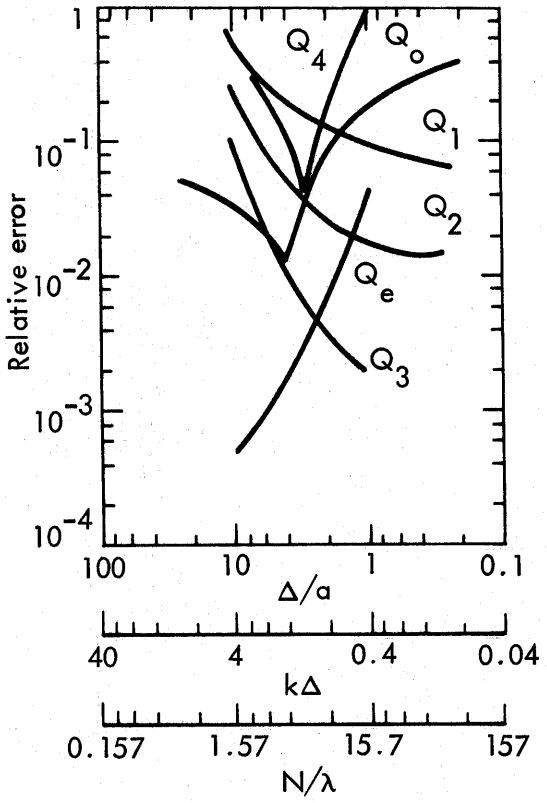

Fig. 7. Relative errors in various Q_i's (self term) for ka = 0.4.

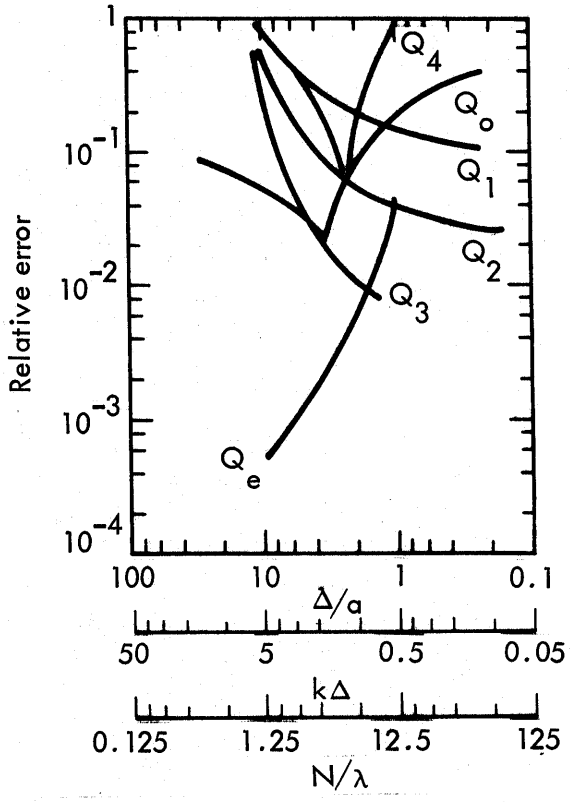

Fig. 8. Relative errors in various Q_i's (self term) for ka = 0.5.

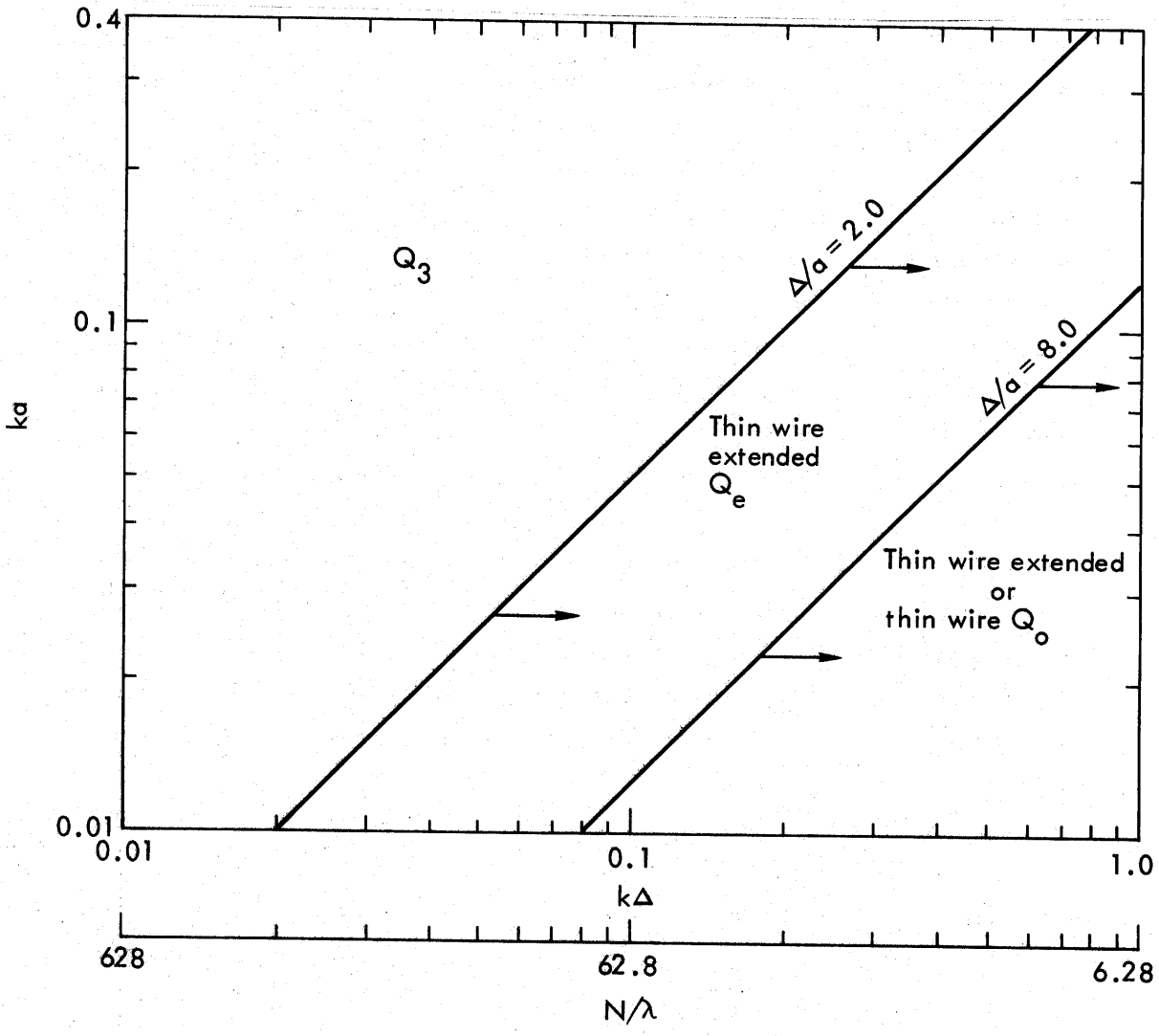

Fig. 9. Representations that lead to self-term errors less than 1%.

RELATIVE ERRORS IN THE ADJACENT AND NEXT-TO-ADJACENT TERMS

The various representations used for Q(z) can yield errors even when the interval of integration does not contain the singularity. Because the representations Q_1, Q_2. Q_3, Q_4 are for small displacements, this may not be surprising. However, the user must be aware that difficulties can be encountered. In Appendix A, we show that sizeable errors in adjacent terms can arise by using the thin-wire kernel $G_0(z,z')$. We now investigate a similar occurrence in representations of Q(z). In this portion of the study we include only those representations appearing in Fig. 9, i.e., Q_0, Q_e, Q_3.

Figure 10 shows the range of relative errors induced by the various approximations as a function of Δ/a with the parameter ka varying from 10^{-4} to 10^{-2}. The relative error is almost insensitive to the variation. However, Figs. 11 through 15 show the relative errors for Q_0 and Q_3, as well as the relative constancy of the relative error in Q_e as ka increases. The results presented in Figs. 10 through 15 allow us to plot the range of applicability for the various representations when evaluating the adjacent terms (Fig. 16). Finally, because we have considered the adjacent terms, it is appropriate to consider the next-to-adjacent terms, $\alpha = z + 3\Delta/2$, and $\beta = z + 5\Delta/2$, Having performed a similar study as that preceding, we present here only the encapsulated result, namely, the plot of the ranges of applicability. Figure 17 depicts the ranges in the ka - kΔ space. It is evident in this plot that the region is enlarged where the extended thin-wire representation provides a relative error less than 1%. Naturally, the region of validity for Q_3 is shrinking because it is a small distance approximation.

The results of our investigation of the various representations of Q(z) are summarized as follows:

- If $\Delta/a > 10.0$, the thin-wire representation can be used everywhere.
- If $\Delta/a > 2.0$, the extended thin-wire representation can be used everywhere.
- If $\Delta/a < 2.0$, Q_3 should be used for the self and adjacent terms and for the next-to-adjacent if $\Delta/a < 1.0$.
- For $1.0 \leq \Delta/a \leq 2.0$ Q_3 is used for the self and adjacent terms and the extended thin-wire for the next-to-adjacent term.

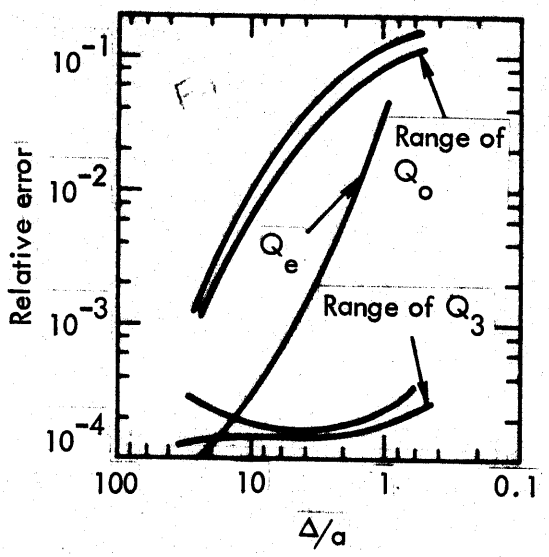

Fig. 10. Relative error in adjacent term for ka in the range 10^{-4} to 10^{-2}.

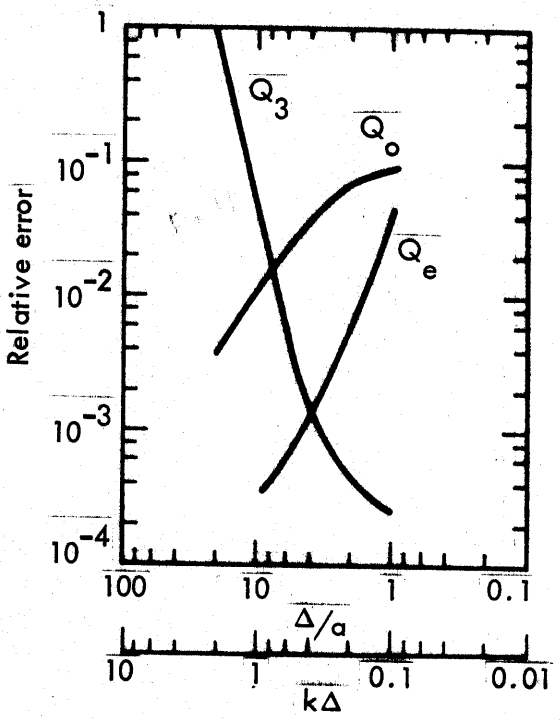

Fig. 11. Relative error in adjacent term for ka = 0.1.

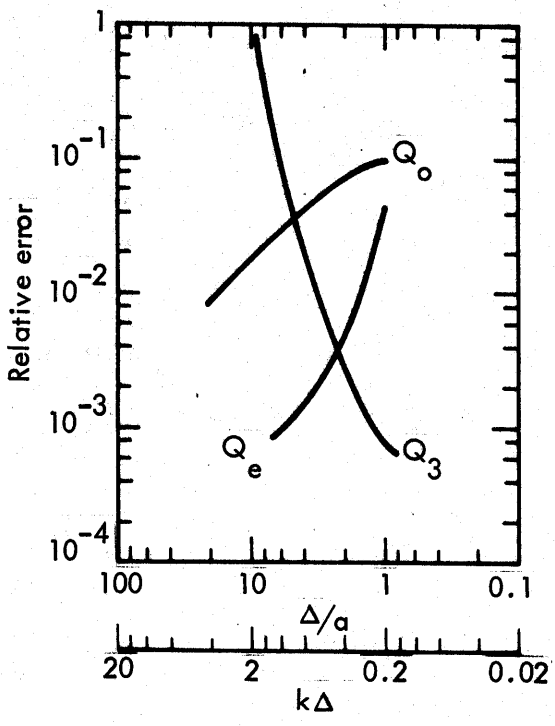

Fig. 12. Relative error in adjacent term for ka = 0.2.

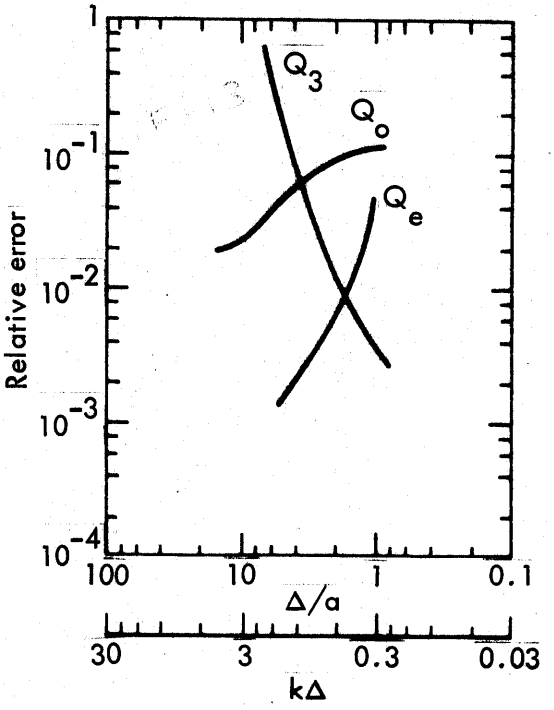

Fig. 13. Relative error in adjacent term for ka = 0.3.

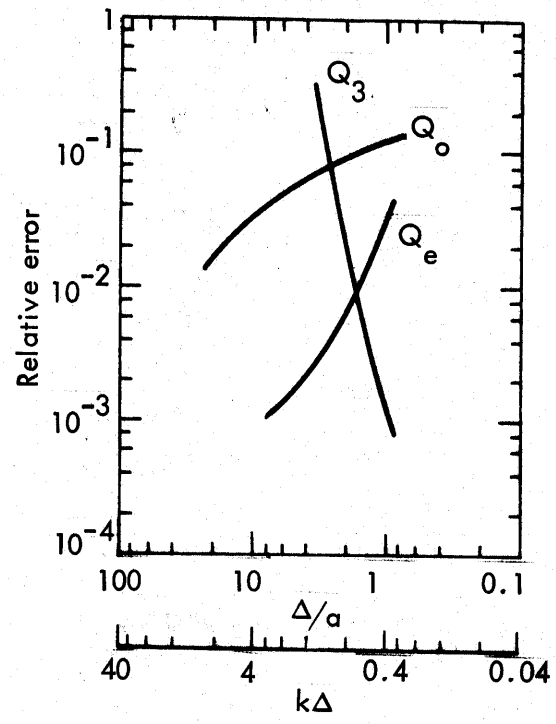

Fig. 14. Relative error in adjacent term for ka = 0.4.

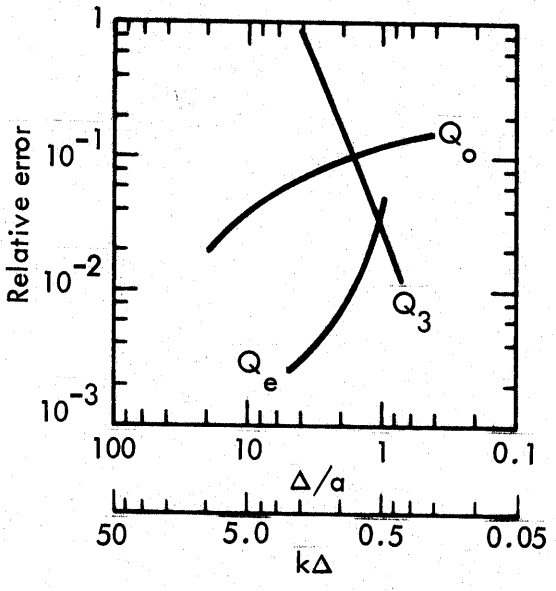

Fig. 15. Relative error in adjacent term for ka = 0.5.

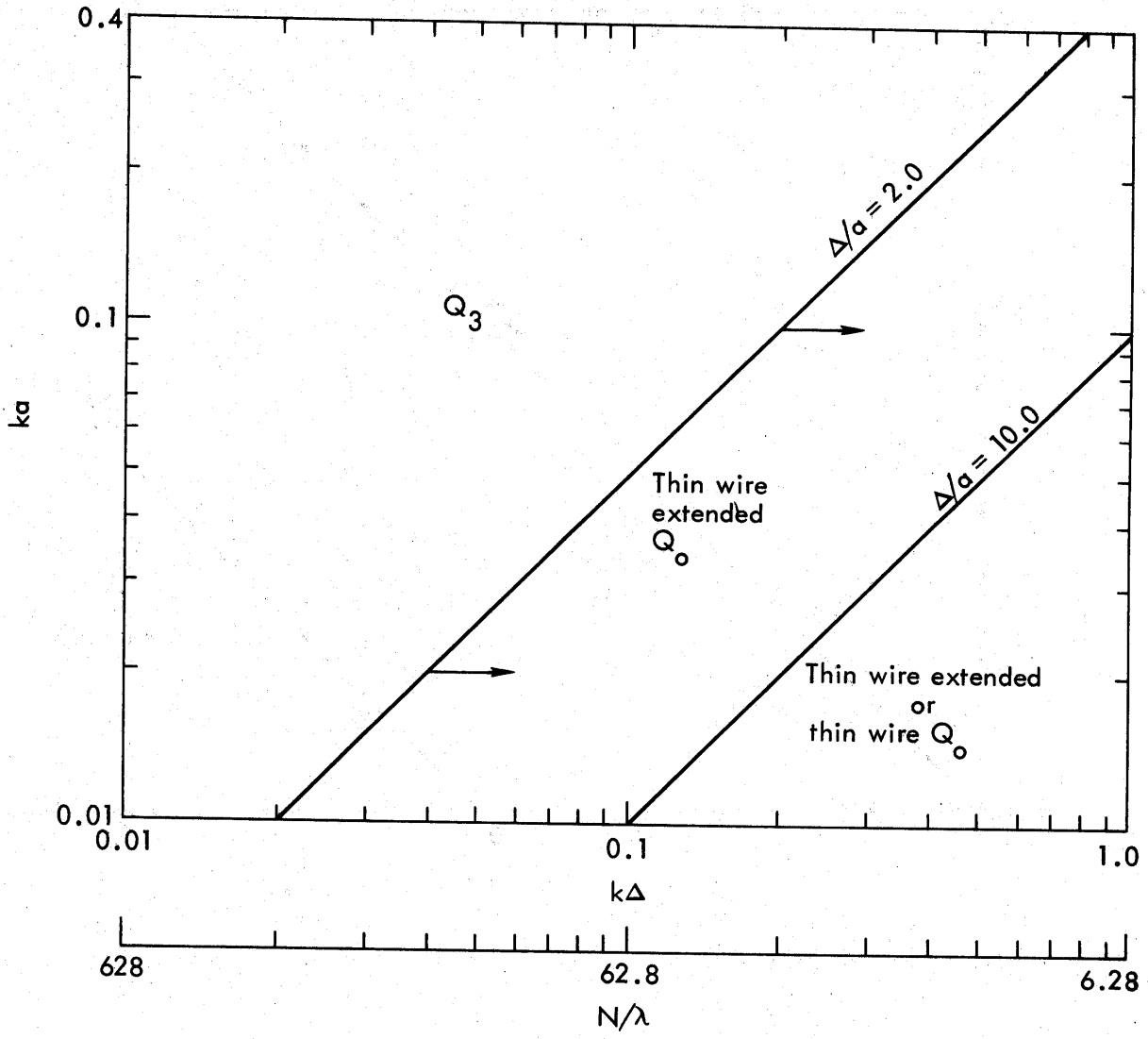

Fig. 16. Representations that lead to adjacent terms errors less than 1%.

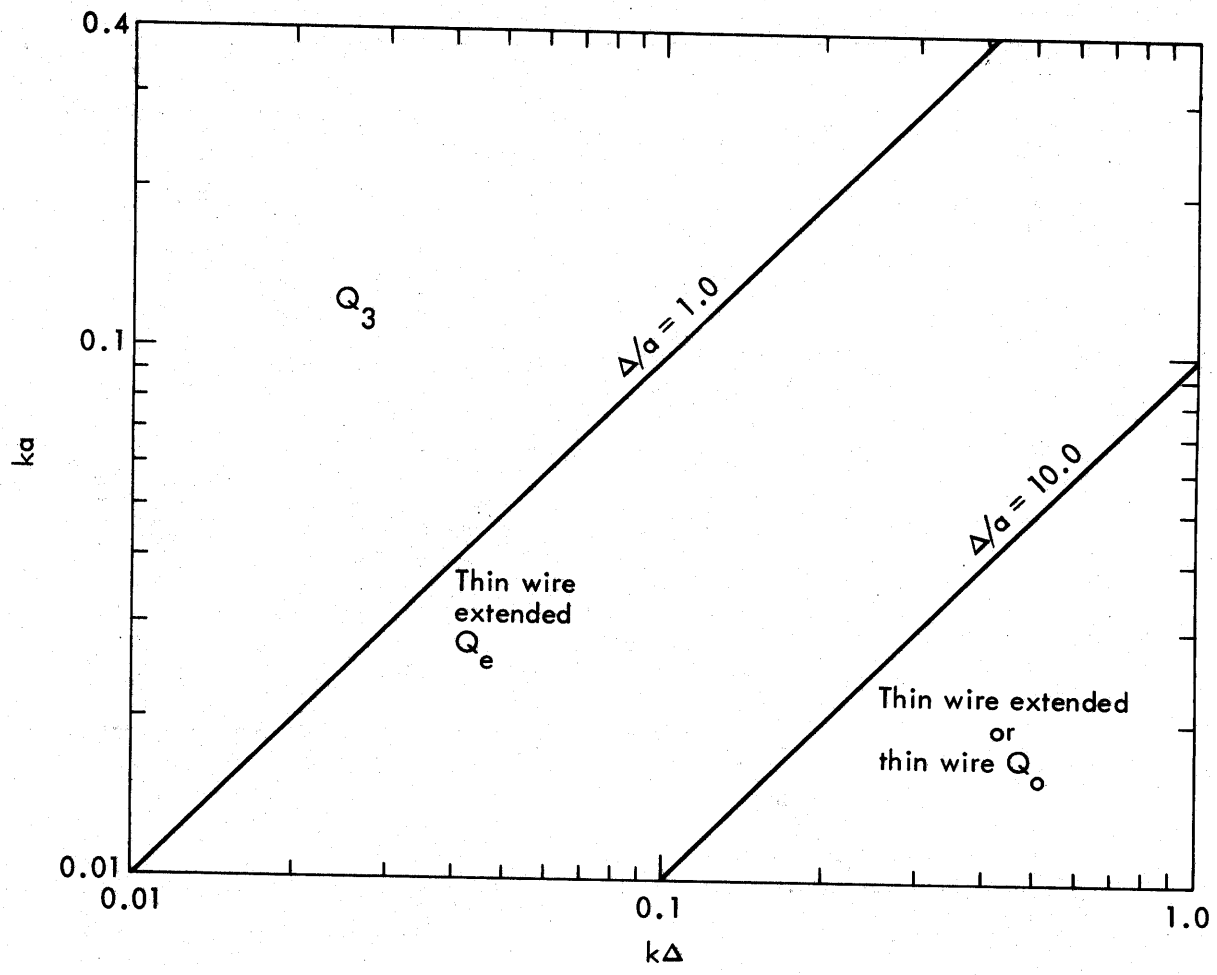

Fig. 17. Representations that lead to next-to-adjacent term errors less than 1%.

REPRESENTATIONS FOR A SINUSOIDAL CURRENT DISTRIBUTION

Several commonly used basis functions for conducting wires include sinusoidal components. Hence, the linear current density in Eq. (3) is written as

$$I(z') = \begin{matrix}\sin \\ \cos\end{matrix} [k(z'-z_i)].$$

However, the axially directed electric field of such a current distribution can be simply written as[1]

$$E_z \sim I(z') \frac{\partial G}{\partial z'}, \left| \begin{matrix}\xi_2 \\ \xi_1\end{matrix} \right. - \frac{\partial I}{\partial z'}, G \left| \begin{matrix}\xi_2' \\ \xi_1\end{matrix} \right. . \tag{35}$$

Hence, it is seen that one requires for calculations only the values of

$$G(z,z') \Big|_\eta = \frac{1}{8\pi^2} \int_0^{2\pi} d\phi \; g(z,\eta,\phi=0,\phi') \tag{36}$$

and

$$\frac{\partial G(z,z')}{\partial z'} \Big|_\eta = \frac{1}{8\pi^2} \int_0^{2\pi} d\phi' \frac{\partial}{\partial z'} g(z,\eta,\phi=0,\phi'), \tag{37}$$

or, in expanded form,

$$G(z,z') \Big|_\eta = \frac{1}{2\pi^2} \int_0^{\pi/2} d\psi \frac{e^{-jkr}}{r} \tag{38}$$

and

$$\frac{\partial G(z,z')}{\partial z'} \Big|_\eta = \frac{1}{2\pi^2} \int_0^{\pi/2} d\psi \frac{jkr+1}{r} \frac{z-\eta}{r} \frac{e^{-jkr}}{r} \tag{39}$$

where

$$r = \sqrt{(z-\eta)^2 + 4a^2 \sin^2 \psi} .$$

These integrals can be performed easily using a numerical scheme as there is no singularity. Also, to circumvent the numerical integration, one can use approximate forms. For instance, for large $|z - z'|$, $r = \sqrt{(z-\eta)^2 + a^2}$,

$$G(z,z')\Big|_{\eta} \approx \frac{1}{4\pi} \frac{e^{-jkr}}{r}, \tag{40}$$

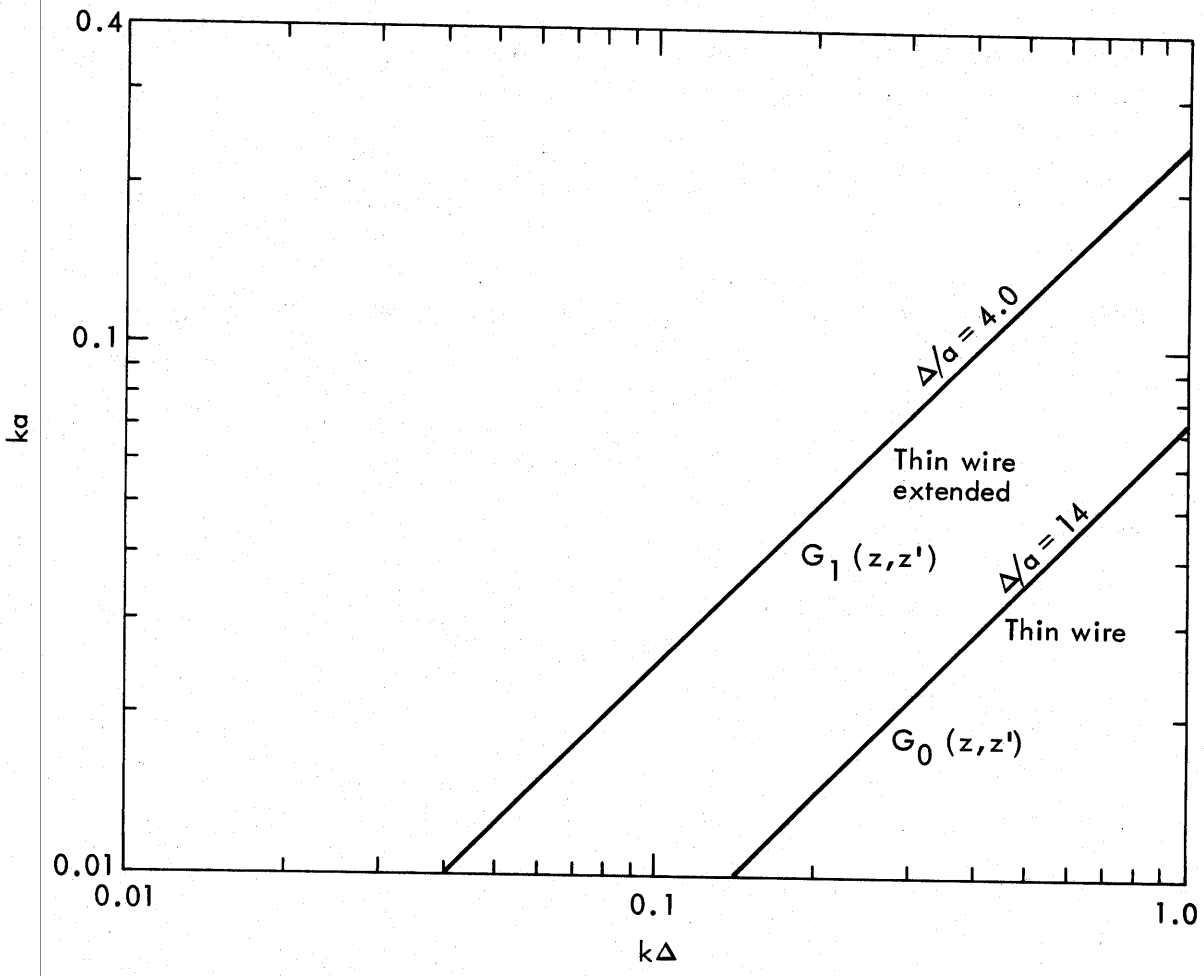

Fig. 18. Representations leading to kernel evaluation with less than 1% error.

and

$$\frac{\partial G(z,z')}{\partial z'}\Big|_{\eta} = \frac{1}{4\pi} \frac{jkr+1}{r} \frac{z-\eta}{r} \frac{e^{-jkr}}{r}. \tag{41}$$

For small $|z - z'|$ the approximations in Appendix A can be used. Naturally, due regard must be paid in both cases to the errors involved in these approximations.

Figure 18 shows the region for which the relative error realized in approximating $G(z-z' = \Delta/2)$ [Eq. (37)] by the thin-wire and extended thin-wire representations are less than 1%. A similar plot is provided in Fig. 19 for the derivative at the end of the interval.

It can be concluded that $G(z,z')$ as given by Eq. (36) can be evaluated numerically for $\Delta/a < 4.0$, by using the extended thin-wire version for $\Delta/a > 4.0$, and the thin-wire version for $\Delta/a > 14.0$. For the derivative term, these limits become $\Delta/a > 4.0$ and 30.0, respectively.

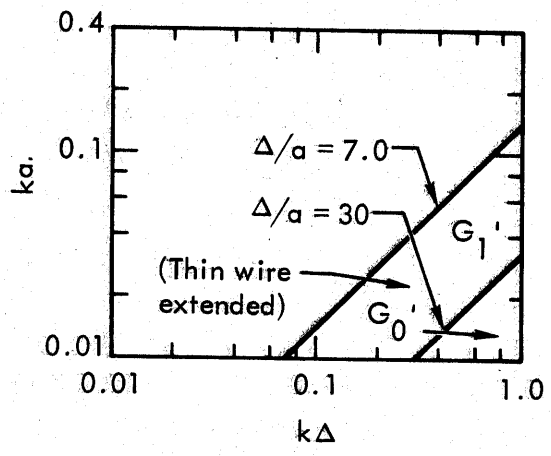

Fig. 19. Representations leading to derivitive evaluation with less than 1% error.

SECTION IV
Conclusions

The widely used thin-wire representation has been shown to be deficient, particularly when displacements from the singularity are small. Alternative representations have been provided and regions of validity have been delineated. Constant and sinusoidal representations for current have been considered and the errors in evaluating self terms, adjacent terms and next-to-adjacent terms have been evaluated. Using the results contained here, a user can employ the form that is consistent with his accuracy requirements, yet, in most cases, without having to numerically evaluate the integrals containing singularities.

Appendix A. A Classical Approach to the Kernel Approximation

Introduction

An integro-differential equation for the current distribution I(z) on a cylindrical dipole antenna of length 2L and radius a with general but azimuthally symmetric excitation $E_z^i(z)$, as given by Ref. 1, is

$$\left(\frac{\partial^2}{\partial z^2} + k^2\right) \int_L^L I(z')\, G(z,z',a)\, dz' = -j\omega\varepsilon\, E_z^i(z), \qquad \text{(A-1)}$$

where the kernel G(z,a',a), because of the assumed symmetry, has the exact representation

$$G(z,z',a) = \frac{1}{8\pi^2 a} \int_0^{2\pi} \frac{e^{-jka\sqrt{4\sin^2\frac{\psi'}{2} + \frac{z-z'^2}{a}}}}{\sqrt{4\sin^2\frac{\psi'}{2} + \frac{z-z'^2}{a}}}\, d\psi'. \qquad \text{(a-2)}$$

The characteristics of G(a,a',a) near its singularity ($\psi' = 0$, $z = z'$) play an important role in establishing the solution for I(z). The solution of Eq. (A-1) or alternative forms (e.g., Hallen's integral equation) by approximate numerical techniques, such as point matching (discretization and collocation),[8,9] requires (for efficiency) the knowledge of the valid approximations to G(z,z',a) that do not require aψ' integration. Also, the ranges of validity of the approximations must be known and observed to ensure accurate solutions. We derive here various representations and graphically compare them to establish their respective regions of validity. An error that arises when a particular approximation is used will be considered.

DERIVATION OF APPROXIMATIONS

The kernel G(z,z',a) can be written as the sum of two integrals:[1] one containing a phase-stationary, singular integrand, and the other, a regular integrand:

$$G(z,z',a) = \frac{1}{8\pi^2} \left\{ \int_0^{2\pi} \frac{d\psi'}{r} - \int_0^{2\pi} \frac{1 - e^{-jkr}}{r}\, d\psi' \right\},$$

[8] K. K. Mei, IEEE Trans. Ant. and Propag. AP-13, 374 (1965).

[9] S. A. Schelkunoff, Advanced Antenna Theory (John Wiley and Sons, New York, 1952).

where

$$r = \sqrt{4a^2 \sin^2 \frac{\psi'}{2} + (z-z.)^2} \ .$$

By performing a simple transformation and substituting a Maclaurin series about kr = 0 for the exponential and integrating the terms to order k^3r^3 in the integrand, the kernel can be written as

$$G(z,z',a) = \frac{1}{8\pi^2 a}\left[2k_1K(k_1) - 2\pi jka - \frac{4k^2a^2}{k_1}E(k_1) - \right.$$

$$\left. \frac{2\pi}{3} jk^3a^3 + \frac{\pi}{3} jk^3a\,(z-z')^2 + \ldots\right] , \qquad \text{(a-3)}$$

where

$$k_1 = 2a \,/\, \sqrt{4a^2 + (z-z')^2} \ ,$$

and $K(k_1)$ and $E(k_1)$ are complete elliptic integrals of the first and second kind, respectively. Equation (A-3) is an exact expression for the kernel when all the terms in the series are included. However, for ease of computation, we consider various approximations for the kernel for limited ranges of the variable $|z-z'|/a$.

An approximation valid for kr << 1 results when the first two terms of Eq. (A-3) are retained. This approximation, referred to as G_{a1}, contains the result of the first integral and the integral pertaining to n = 1 in the Maclaurin series. A simplification of G_{a1} can be realized by making use of the first term of the series expansion for $K(k_1)$,[10] viz.,

$$K(k_1) = \ln \sqrt{1 - k_1^2} \ .$$

For a small $|z-z'|/a$, we have $1 - k_1^2 \approx |z-z'|/2a$, so that a second approximation is obtained:

$$G_{a2} = \frac{1}{8\pi^2 a}\left\{2k_1 \ln \frac{8a}{|z-z'|} - 2\pi jka\right\} , \qquad \text{(A-4)}$$

[10] E. Jahnke and F. Emde, Tables of Functions (Dover Publications, New York, 1945).

and because $k_1 \approx 1$, a third approximation follows:

$$G_{a3} = \frac{1}{8\pi^2 a}\left\{2\ \ell n\ \frac{8a}{|z-z'|} - 2\pi jka\right\}. \qquad \text{(A-5)}$$

Let G_b be the kernel that results from keeping the first three terms of Eq. (A-3), i.e., up to the n = 2 term in the Maclaurin series. This approximation can be written as

$$G_b = \frac{1}{8\pi^2 a}\left\{2k_1 K(k_1) - 2\pi jka - \frac{4k^2a^2}{k_1}\ E(k_1)\right\}. \qquad \text{(A-6)}$$

The approximations G_{a1}, G_{a2}, G_{a3}, and G_b have been written for $kr << 1$. From the definition of r, we see that this requires $ka\sqrt{4 + (z-z'/a)^2} << 1$. For $ka << 1$, the inequality may be satisfied for quite large values of $|z-z'|/a$.

The kernel can also be approximated over the remainder of the range of $|z-z'|/a$. One such approximation, which is generally referred to as the thin-wire kernel, is obtained by a physical approximation in which the source point z' is considered to be on the surface of the cylinder and the observation point z to be on the axis of the cylinder, so that the approximate kernel can be written as

$$G_d = \frac{1}{8\pi^2 a}\left[2\pi\ \frac{e^{-jka\sqrt{1+\left(\frac{z-z'}{a}\right)^2}}}{\sqrt{1+\left(\frac{z-z'}{a}\right)^2}}\right]. \qquad \text{(A-7)}$$

Another approximation can be derived by letting $|z-z'|/a >> 4\sin^2\psi'/2$ for all ψ' in Eq. (A-2). Then,

$$G_e = \frac{1}{8\pi^2 a}\left(2\pi\ \frac{e^{-jka\left|\frac{z-z'}{a}\right|}}{\left|\frac{z-z'}{a}\right|}\right). \qquad \text{(A-8)}$$

This kernel is identical to the one that results from placing both source and observation points on the axis of the cylinder. It can also be derived from Eq. (A-7) by requiring that $|z-z'/a| >> 1$.

COMPARISON OF APPROXIMATIONS TO THE KERNEL

The various approximations to the exact kernel G (normalized to $8\pi^2 a$) are plotted as a function of normalized displacement in Figs. A-1 and A-2. The ψ' integration indicated in Eq. (A-2) was performed numerically to establish the regions in which the approximations most accurately represent the exact kernel. The figures are for radius-to-wavelength ratios (a/λ) of 10^{-4} and 10^{-2}, respectively, and show the dependence of the regions of validity on the dipole thickness.

Figures (A-1) and (A-2) show that a single approximation, adequate for the entire range of $|z-z'|/a$, does not exist. They also indicate that G_{a1}, G_b, and G_{a3} are excellent approximations for small values of the normalized displacement $|z-z'|/a$. It is also evident that the accuracy of the approximations, especially for the imaginary part, deteriorates as the ratio

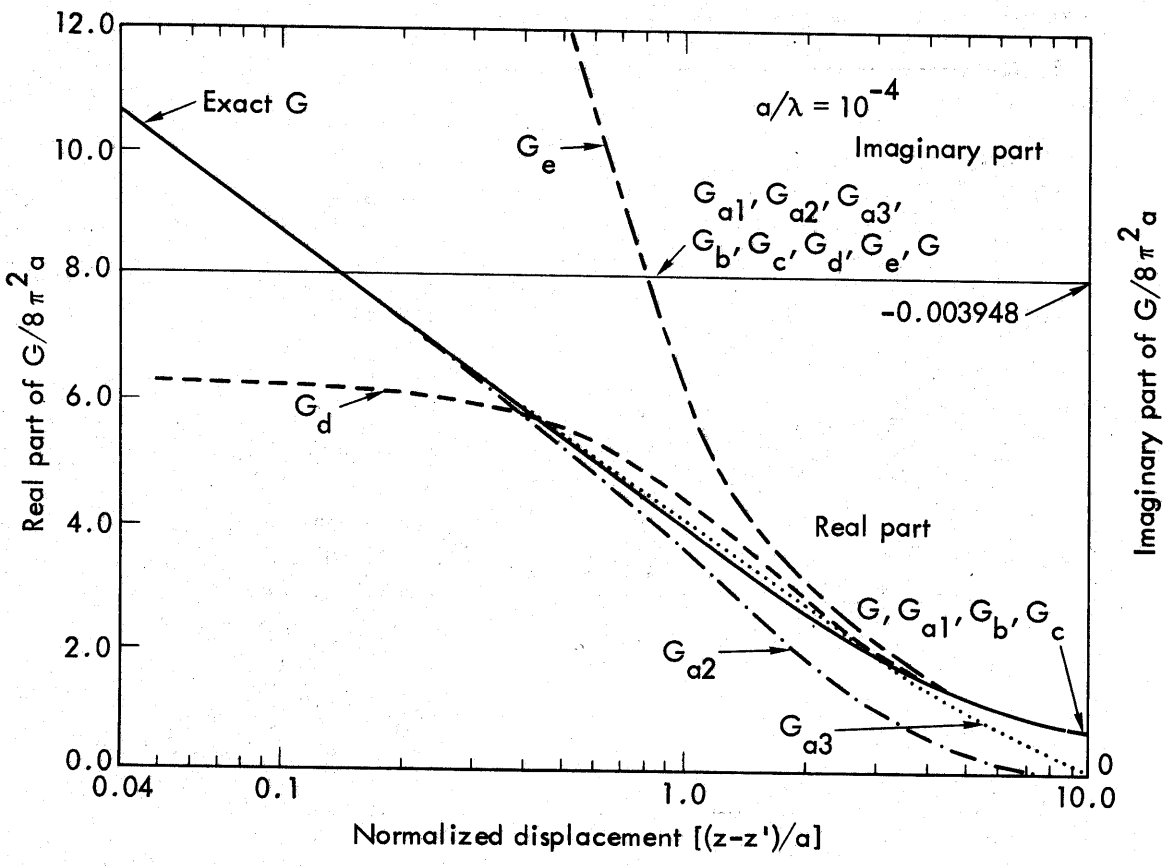

Fig. A-1. A comparison of various approximations to the kernel for $a/\lambda = 10^{-4}$.

a/λ increases. Because it is necessary to use different approximations over various ranges of $|z-z'|/a$, one can, in view of the simplicity of the expressions, choose

$$G(z,z',a) = \frac{1}{8\pi^2 a} \begin{cases} 2\ \ell n \dfrac{8a}{|z-z'|} - 2\pi jka & |z-z'|/a < 0.3 \\ 2\ k_1 K(k_1) - 2\pi jka & |z-z'|/a \leq 3.0 \\ 2\pi \dfrac{e^{-jka\sqrt{1+\left(\frac{z-z'}{a}\right)^2}}}{\sqrt{1+\left(\frac{z-z'}{a}\right)^2}} & |z-z'|/a > 3.0 \ . \end{cases}$$

The first entry is included because it has an analytically integrable singularity at $z = z'$. The approximation G_e serves well for $|z-z'|/a > 6.0$. The restriction on the radius-to-wavelength ratio should be observed when establishing these regions of validity.

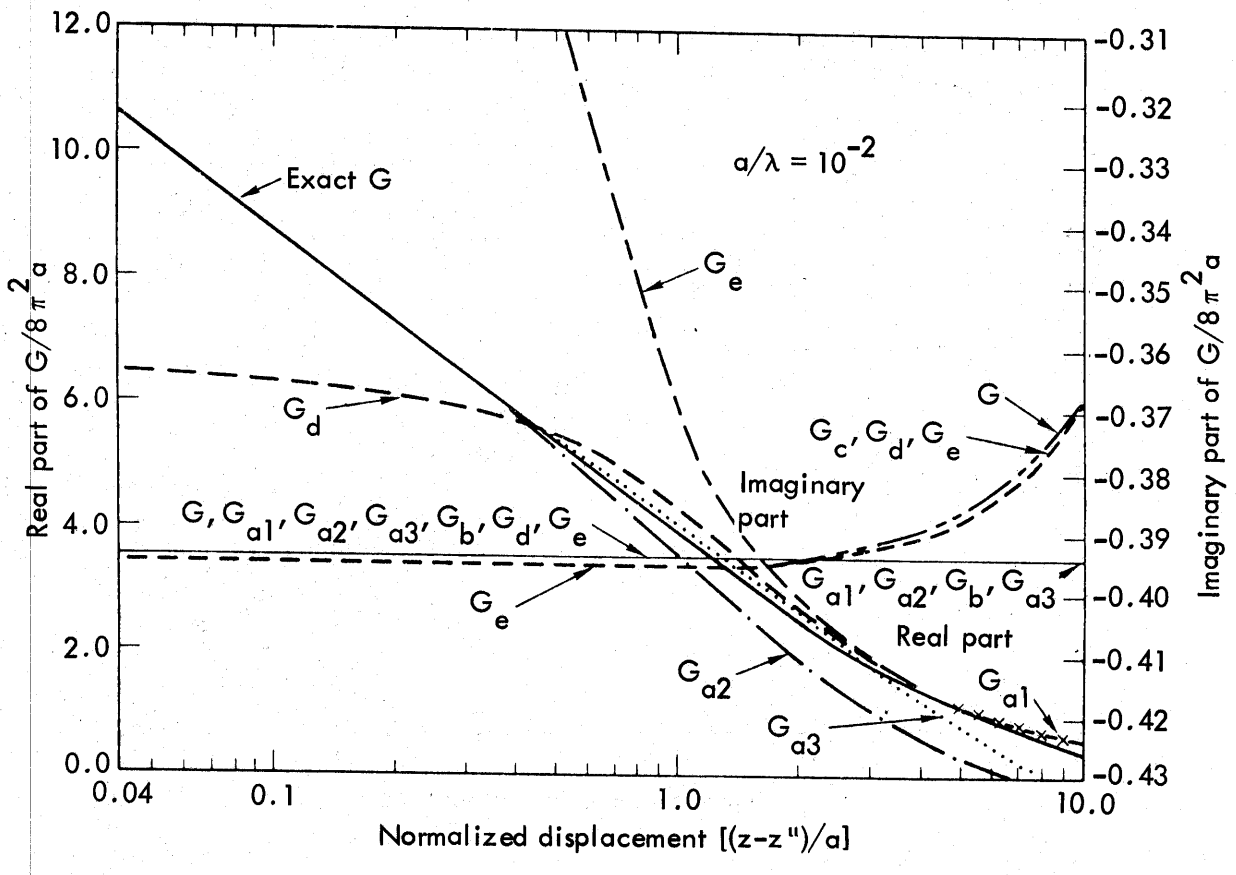

Fig. A-2. A comparison of various approximations to the kernel for $a/\lambda = 10^{-2}$.

A CRITICISM OF THE THIN-WIRE KERNEL

Although the thin-wire kernel G_d has generally been used over the entire range, this approximation suffices only for $|z-z'|/a < 3.0$. For normalized distances less than 3.0, this kernel is clearly deficient because its real part does not exhibit the proper singularity. However, because we are generally concerned with its integral rather than its functional value, the thin-wire kernel can be used over a wider range than Figs. A-1 and A-2 indicate.

It is interesting to consider the relative error that arises when the thin-wire kernel is used in evaluating the integral on the right hand side for $z_i = z_j$. Figure A-3 plots the relative error given by

$$\varepsilon_1 = \frac{\int_{z_j-\Delta/2}^{z_j+\Delta/2} G_d(z_j,z',a)dz' - \int_{z_j-\Delta/2}^{z_j+\Delta/2} G(z_j,z',a)dz'}{\left|\int_{z_j-\Delta/2}^{z_j+\Delta/2} G(z_j,z',a)dz'\right|}$$

vs the normalized interval half-length $\Delta/2a$. Also plotted is the relative error ε_2 for evaluating the integration over the adjacent interval, $z_j + \Delta/2 \leq z_i + 3\Delta/2$. In the matrix solution of the integral equation, these plots of the relative error in the diagonal and first-off diagonal terms in the coefficient matrix. The relative error in the diagonal and first-off diagonal elements can be unacceptably large for small intervals ($\Delta/a<2.0$). The ε_2 is not a monotonically decreasing function, but rather suffers a sharp minimum when the interval contains the crossover of G_d and G at $|z-z'|/a \approx 0.4$. The relative errors (ε_1 and ε_2) can be reduced by increasing the interval

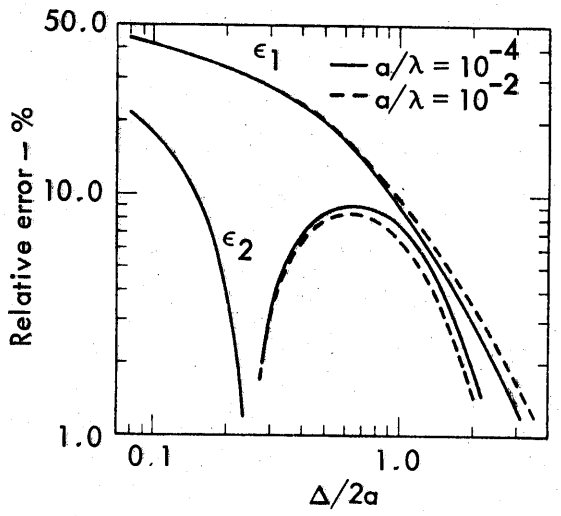

Fig. A-3. Relative errors ($\varepsilon 1$ and $\varepsilon 2$) resulting from the use of the thin-wire kernel.

length. For intervals on the order of six radii ($\Delta/a \approx 6.0$), the relative errors are less than 2%.

CONCLUSIONS

Various approximations for the exact kernel G have been compared and a deficiency in the thin-wire kernel pointed out. A consideration of the errors involved in the integral of the kernel leads to the conclusion that the thin-wire kernel should be used only for sufficiently large intervals ($\Delta/a \lesssim 6.0$). A matrix solution of the integral equation for the dipole using G_d might not converge as the number of segments N increases indefinitely, but rather might begin to diverge beyond a certain number N_0. To increase the number of segments, other approximations for the kernel near the singularity must be used.

Appendix B. An Alternative Approach to the Kernel Approximation

An alternate representation has been provided by Tesche.[7] There, Q(z) is written as

$$Q(z) = \frac{1}{8\pi^2} \int_0^{2\pi} d\phi' \; P(z,\phi'), \tag{B-1}$$

where

$$P(z,\phi') = \int_\alpha^\beta dz' \; g(z,z',\phi = 0,\phi'),$$

Expanding on his derivation, with $R = \sqrt{4a^2 \sin^2 \frac{\phi'}{2} + (z-z')^2}$, we obtain

$$P(z,\phi') = - \sum_{m=0}^{\infty} \frac{(-jk)^m}{m!} \int_{z-\alpha}^{z-\beta} d\xi \left(4a^2 \sin^2 \frac{\phi'}{2} + \xi^2\right)^{\frac{m-1}{2}}$$

$$= - \ell n \left[\frac{(z-\beta) + \sqrt{4a^2 \sin^2 \frac{\phi'}{2} + (z-\beta)^2}}{(z-\alpha) + \sqrt{4a^2 \sin^2 \frac{\phi'}{2} + (z-\alpha)^2}} \right] - jk(\beta-\alpha)$$

$$- \sum_{m=2}^{\infty} \frac{(-jk)^m}{m!} \int_{z-\alpha}^{z-\beta} d\xi \; R^{m-1} \; . \tag{B-2}$$

The first two terms are independent of ka, as in the previous form, and the first term is static. Tesche uses these two terms in his approximate representation, in which he sets $\alpha = z - \Delta/2$ and $\beta = z + \Delta/2$, and obtains

$$P(z,\phi') = -jk\Delta - 2 \ell n \left[\frac{\Delta}{4a} + \sqrt{\sin^2 \frac{\phi'}{2} + \left(\frac{\Delta}{4a}\right)^2} \right] - 2 \ell n \left[\sin \frac{\phi'}{2} \right] . \tag{B-3}$$

Hence, the first-order approximation to Q(z) becomes

$$Q_1(z) = \frac{-jk\Delta}{4\pi} + \frac{1}{2\pi}\,\ell n\ 2 + \frac{1}{4\pi^2}\int_0^{2\pi} d\phi'\ \ell n\left[\frac{\Delta}{4a} + \sqrt{\sin^2\frac{\phi'}{2} + \left(\frac{\Delta}{4a}\right)^2}\right], \quad \text{(B-4)}$$

which is again, as expected, independent of ka but dependent on $k\Delta$.

Let us now write Q(z) with higher-order terms included, i.e.,

$$Q(z) = -jk\,\frac{\beta-\alpha}{4\pi} - \frac{1}{8\pi^2}\int_0^{2\pi} d\phi'\ \ell n\left[\frac{(z-\beta) + \sqrt{4a^2\sin^2\frac{\phi'}{2} + (z-\beta)^2}}{(z-\alpha) + \sqrt{4a^2\sin^2\frac{\phi'}{2} + (z-\alpha)^2}}\right]$$

$$- \sum_{m=2}^{\infty} \frac{(-jk)^m}{m!}\,\frac{1}{8\pi^2}\int_0^{2\pi} d\phi' \int_{z-\alpha}^{z-\beta} d\xi\ R^{m-1}\ . \quad \text{(B-5)}$$

Thus, including terms up to order pertaining to m = 2, we obtain a second-order approximation

$$Q_2(z) = -jk\,\frac{\beta-\alpha}{4\pi} - \frac{1}{8\pi^2}\int_0^{2\pi} d\phi' \left\{\left[1 - k^2a^2\sin^2\frac{\phi'}{2}\right]\ell n\left[\frac{\frac{z-\beta}{2a} + \sqrt{\sin^2\frac{\phi'}{2} + \left(\frac{z-\beta}{2a}\right)^2}}{\frac{z-\alpha}{2a} + \sqrt{\sin^2\frac{\phi'}{2} + \left(\frac{z-\alpha}{2a}\right)^2}}\right]\right.$$

$$\left. - \frac{1}{2}\left[k^2(z-\beta)a\sqrt{\sin^2\frac{\phi'}{2} + \left(\frac{z-\beta}{2a}\right)^2} - k^2(z-\alpha)a\sqrt{\sin^2\frac{\phi'}{2} + \left(\frac{z-\alpha}{2a}\right)^2}\right]\right\}, \quad \text{(B-6)}$$

which has terms of order ka and $(ka)^2$.

For $\alpha = z - \Delta/2$ and $\beta = z + \Delta/2$, we have

$$Q_2(z) = \frac{-jk\Delta}{4\pi} - \frac{1}{2\pi^2}\int_0^{\pi/2} d\psi \left\{[1 - k^2a^2\sin^2\psi]\ \ell n\left[\frac{-\frac{\Delta}{4a} + \sqrt{\sin^2\psi + \left(\frac{\Delta}{4a}\right)^2}}{\frac{\Delta}{4a} + \sqrt{\sin^2\psi + \left(\frac{\Delta}{4a}\right)^2}}\right]\right.$$

$$\left. + \frac{1}{2}\,(k\Delta)\,(ka)\sqrt{\sin^2\psi + \left(\frac{\Delta}{4a}\right)^2}\right\}. \quad \text{(B-7)}$$

The last term can be analytically integrated:[5]

$$\int_0^{\pi/2} d\psi \sqrt{\sin^2\psi + \left(\frac{\Delta}{4a}\right)^2} = \sqrt{1 + \left(\frac{\Delta}{4a}\right)^2}\, E\left(\pi/2, \frac{1}{\sqrt{1 + \left(\frac{\Delta}{4a}\right)^2}}\right), \qquad \text{(B-8)}$$

where $E(\pi/2,k')$ is an elliptic integral of the second kind, so that

$$Q_2(z) = \frac{-jk\Delta}{4\pi} - \frac{1}{4\pi^2}(k\Delta)(ka)\sqrt{1 + \left(\frac{\Delta}{4a}\right)^2}\, E\left(\frac{1}{\sqrt{1 + \left(\frac{\Delta}{4a}\right)^2}}\right) \qquad \text{(B-9)}$$

$$+ \frac{1}{\pi^2}\int_0^{\pi/2} d\psi \left[1 - k^2a^2\sin^2\psi\right]\left\{\ell n\left[\frac{\Delta}{4a} + \sqrt{\sin^2\psi + \left(\frac{\Delta}{4a}\right)^2}\right] - \ell n \sin\psi\right\}.$$

It is well known that

$$\int_0^{\pi/2} \ell n\,(\sin\psi)\, d\psi = -\pi/2\ \ell n 2$$

and that

$$\int_0^{\pi/2} \sin^2\psi\ \ell n\,(\sin\psi)\, d\psi = \frac{\pi}{8}(1 - 2\ \ell n\ 2).$$

Then, we have

$$Q_2(z) = \frac{-jk\Delta}{4\pi} + \frac{\ell n 2}{2\pi}\left[1 - \frac{(ka)^2}{2}\right] + \frac{1}{8\pi}(ka)^2 - \frac{1}{4\pi^2}(k\Delta)(ka)\sqrt{1 + \left(\frac{\Delta}{4a}\right)^2}\, E\left(\frac{1}{\sqrt{1 + \left(\frac{\Delta}{4a}\right)^2}}\right)$$

$$+ \frac{1}{\pi^2}\int_0^{\pi/2} d\psi\,(1 - k^2a^2\sin^2\psi)\,\ell n\left[\frac{\Delta}{4a} + \sqrt{\sin^2\psi + \left(\frac{\Delta}{4a}\right)^2}\right] + O(kR). \qquad \text{(B-10)}$$

We now introduce the m = 3 term in Eq. (B-2), which can be written as

$$I_3 = \frac{(-jk)^3}{m!}\frac{1}{8\pi^2}\int_0^{2\pi} d\phi' \int_{z-\alpha}^{z-\beta} d\xi \left(4a^2 \sin^2\frac{\phi'}{2} + \xi^2\right) = \frac{-jk^3}{6}\left\{\frac{\Delta a^2}{2\pi} + \frac{\Delta^3}{6\pi}\right\}, \quad \text{(B-11)}$$

so that

$$Q_3(z) = \frac{-jk\Delta}{4\pi}\left[1 - \frac{k^2a^2}{3} - \frac{k^2\Delta^2}{72}\right] + \frac{\ell n 2}{2\pi}\left[1 - \frac{k^2a^2}{2}\right]$$

$$+ \frac{1}{8\pi}(ka)^2 - \frac{1}{4\pi^2}(k\Delta)(ka)\sqrt{1 + \left(\frac{\Delta}{4a}\right)^2}\; E\,\frac{1}{\sqrt{1 + \left(\frac{\Delta}{4a}\right)^2}}$$

$$+ \frac{1}{\pi^2}\int_0^{\pi/2} d\psi\,(1 - k^2a^2\sin^2\psi)\,\ell n\left[\frac{\Delta}{4a} + \sqrt{\sin^2\psi + \left(\frac{\Delta}{4a}\right)^2}\right]. \quad \text{(B-12)}$$

This expression should be more than adequate for most self-term evaluations. The required integral is only slightly more complicated than that given by $Q_1(z)$.

References

1. J. A. Stratton, Electromagnetic Theory, (McGraw Hill, New York, 1941).

2. S. A. Schelkunoff and H. T. Friis, Antenna Theory and Practice, (John Wiley and Sons, Inc., New York, 1952).

3. A. J. Poggio, and P. E. Mayes, Numerical Solution of Integral Equations of Dipole and Slot Antennas Including Active and Passive Loading, Air Force Avionics Laboratory Rept., Wright-Patterson Air Force Base, Ohio (1969).

4. F. M. Tesche, Evaluation of the Surface Integral Occurring in the E-Field Integral Equations for Thin-Wire Antennas, Air Force Weapons Laboratory, Kirtland Air Force Base, New Mexico, Mathematics Note 29 (1973).

5. I. S. Gradshteyn and I. M. Ryzhik, Tables of Integrals, Series, and Products (Academic Press, New York, 1965).

6. E. Hallen, Electromagnetic Theory (John Wiley and Sons, 1962).

7. R. F. Harrington, Field Computation by Moment Methods (MacMillan, 1968).

8. K. K. Mei, IEEE Trans. Ant. And Propag. AP-13, 374 (1965).

9. S. A. Schelkunoff, Advanced Antenna Theory (John Wiley and Sons, New York, 1952).

10. E. Jahnke and F. Emde, Tables of Functions (Dover Publications, New York, 1945).