Mathematics Notes

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Suboptimal System Approximation/Identification With Know Error

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Abstract

This paper presents a noniterative method for approximating empirical signals over $[0,\infty)$ by a linear combination of exponentials. The technique results in a suboptimal approximation. Notably, the dependence of the suboptimal exponents s_i' on the integral square error ε is such that $\lim\ (\varepsilon \to 0)$ $s_i' = s_i$, the optimal exponents. The method may also be used for system identification. It is especially useful when the system is modelled by a black box and one has access only to the input and output terminals of the system. A technique is demonstrated to find the multiple poles of a system along with the residues at the poles when the output of the system to a known input is given. Among the advantages of the method are its natural insensitivity to noise in the data and the explicit determination of the signal order. Representative computations are made of the poles from the transient response of a conducting pipe tested at the ATHAMAS-I EMP simulator.

PREFACE

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TABLE OF CONTENTS

LIST OF FIGURES

SECTION I

INTRODUCTION

In recent years the singularity expansion method (SEM) [1, 2] has been used very successfully to study transient phenomena in electromagnetic radiation and scattering problems. With this approach, information is obtained about the electromagnetic structures from measured transient responses to known inputs. The information leads to a characterization of the impulse response of the electromagnetic system by a sum of damped exponentials. It is desirable to know the complex natural frequencies or poles with a high degree of accuracy. The problem of extraction of the poles from the measured transient response data is reduced to a system approximation/identification problem.

The field of approximation/identification has developed rapidly during the last decade. A good summary of the developments is available in [3]. It also includes 230 references. The use of exponential basis functions to approximate real time signals has been extensively studied by several investigations. The paper by

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1 C. E. Baum, "Emerging Technology for Transient and Broadband Analysis and Synthesis of Antennas and Scatterers," Interaction Notes - 300, Nov. 1976 (Also in Proc. IEEE, Vol. 64, No. 11, Nov. 1976, pp. 1598-1976.

2 C. E. Baum, "Singularity Expansion Method," in L. B. Felsen Ed., Transient Electromagnetic Fields," New York: Springer Verlag 1976.

3 K. J. Astrom and P. Eykhoff, "System Identification - A Survey," Automatica, 1971, pp. 123-162.

McDonough and Huggins [4] is of particular interst because it describes a procedure for obtaining the optimum exponents and weighting constants needed to minimize the integrated square error, ε. Unfortunately, this approach requires the solution of a set of nonlinear simultaneous equations in an iterative manner.

Prony's algorithm for deriving complex poles and residues for equi-spaced data samples [5] is quite straightforward. However, the Prony Method does not yield an optimum set of poles and residues; nor does it give a measure of accuracy. Also the set of poles obtained by Prony's method depends on how the set of data points was chosen. This is true not only when noise is present but also when the order of approximation is much lower then the actual order of the system. Prony's method becomes quite complicated when the response to a system is given for an arbitrary input and for nonzero initial conditions. Furthermore, when the order of approximation is increased from n to n+1, the whole procedure must be repeated.

The method described in this paper is suboptimal. In this method (first suggested by Jain [6]), the poles are suboptimal. However, they approach the optimum poles [7] when the mean-squared error ε is made small. Having obtained the poles, the weighting constants are evaluated by a least square method. Moreover, the solutions of both approximation and identification problems are unified under this new approach.

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4. R. N. McDonough and W. H. Huggins, "Best Least Squares Representation of Signals by Exponentials," IEEE Trans. Automatic Control, Vol. AC-13 August 1968, pp. 408-412.

5. M. L. Van Blaricum and R. Mittra, "A Technique for Extracting the Poles and Residues of a System Directly from its Transient Response," Interaction Notes - 245, February 1975 (Also in IEEE Trans. on Ant. and Propagat, Vol. AP-23, No. 6, November 1975.

6. V. K. Jain, "Filter Analysis by Use of Pencil Functions: Part I," IEEE Trans. on Circuits and Systems, Vo. CAS-21, No. 5, September 1974.

7. V.K. Jain, "Decoupled Method for Approximation of Signals by Exponentials," IEEE Trans. on Systems Science and Cybernetics, July 1970.

SECTION II

APPROXIMATING A SIGNAL BY EXPONENTIALS

Let the signal $x(t)$ be real for $t \geq 0$ and zero otherwise and be square integrable over $[0,\infty)$. Let the least-square approximation (which minimizes the integrated square error, ε) of $x(t)$ be $g(t)$, which for a given n (the order of approximation) is given by

$$g(t) = \sum_{i=1}^{q} \sum_{j=1}^{m_i} A_{ij} \, t^{(j-1)} \, e^{s_i t} \tag{1}$$

where

$$\sum_{i=1}^{q} m_i = n, \; A_{ij} \neq 0, \; \mathrm{Re}\left[s_i\right] < 0, \; s_i \neq s_j$$

for $i \neq j$. (2)

Here, n equals the total number of poles in the approximation function $g(t)$, m_i is the multiplicity of the ith pole, and q is the number of distinct poles of $g(t)$. If all the system poles are assumed to be distinct, then (1) can be expressed in the simpler form

$$g(t) = \sum_{i=1}^{n} A_i \exp(s_i t) \tag{3}$$

A measure of the accuracy of the approximation of $x(t)$ is the non-negative number ε defined by $x(t) = g(t) + \varepsilon r(t)$, where r satisfies $\langle g,r \rangle = 0$ and $\| g \| = \| r \|$. If the function $x(t)$ is integrated repeatedly n times, a set of $n+1$ functions is obtained, namely

$$\left.\begin{aligned} & x(t) = x_1(t) \\ \text{and} \quad & x_{k+1}(t) = \int_{\infty}^{t} x_k(\tau) \, d\tau, \text{ for } k = 1, \ldots, n \end{aligned}\right\} \tag{4}$$

Also each function can be written as

$x_k(t) = g_k(t) + \varepsilon\, r_k(t)$, where g_k and r_k are obtained by repeated integrations of $g = g_1$ and $r = r_1$, respectively. It is shown in the next section that the set of functions $\{g_1, \ldots, g_{n+1}\}$ forms an n-order subspace S_n of a Euclidean space Ω of functions. When x(t) is in S_n, $\varepsilon = 0$ and the exact exponents can be determined from the n order polynomial equation [7]

$$\sum_{i=1}^{n+1} \sqrt{\Delta_{ii}}\ \lambda^{n+1-i} = 0 \qquad (5)$$

where Δ_{ii} are the positive diagonal cofactors of the matrix $\left[G_{n+1}\right] = \left[g_{ij} = \langle g_i, g_j \rangle\right]$. Moreover, n is the smallest positive integer such that $|G_{n+1}| = 0$. Hence, the calculation of the system exponents s_i is reduced to finding the roots of a polynomial. For the general case (suboptimal case) when $\varepsilon \neq 0$, the positive square root of $|H_{n+1}|$ (where $\left[H_{n+1}\right]$ is defined as $\left[h_{ij} = \langle x_i, x_j \rangle\right]$), is of the same order of magnitude as ε, i.e., $\sqrt{|H_{n+1}|} = \theta(\varepsilon)$ and the diagonal cofactors Δ'_{ii} of $\left[H_{n+1}\right]$ are related to Δ_{ii} as

$$\sqrt{\Delta'_{ii}} = \sqrt{\Delta_{ii}} + 0.5\, \varepsilon\, (\delta_{ii} / \sqrt{\Delta_{ii}}) + \theta(\varepsilon^2) \qquad (6)$$

where δ_{ii} are certain constants, which are dependent only on the function g. To be able to make ε as small as desired one should use a complete set of approximating components. A set of components is called complete relative to a specified class of signals if any member of that class can be approximated with zero integrated squared error, ε (at least, in the limit, as $n \to \infty$). If we consider the class of all square integrable signals (i.e., all signals of finite energy), we find that an infinite number of components are required for a complete

representation (of any member in this class). For the exponential components considered here, it is possible to show by means of Szaz's theorem [8] that the representation will be complete provided

$$\sum_{i=1}^{\infty} \left\{ - \text{Re} \left[s_i \right] / \left[1 + |0.5 + s_i|^2 \right] \right\} \to \infty \qquad (7)$$

As an example, consider the special case when

$$\begin{aligned} \text{Re} \left[s_i \right] &= -\alpha i \\ \text{Im} \left[s_i \right] &= \beta i \end{aligned} \qquad (8)$$

for $\alpha, \beta > 0$. This represents a system where the system poles are the harmonic frequencies of the fundamental $(-\alpha + j\beta)$. Substitution of (8) in (7) results in the representation

$$\sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty} \frac{\alpha i}{1.25 - \alpha i + (\alpha^2 + \beta^2)\, i^2} \qquad (9)$$

Comparison of $\sum_{i=1}^{\infty} a_i$ with the series $\sum_{i=1}^{\infty} \frac{\alpha}{\alpha^2 + \beta^2} \cdot \frac{1}{i}$

which diverges, reveals that indeed the series $\sum_{i=1}^{\infty} a_i$ diverges since $a_i > \frac{\alpha}{(\alpha^2 + \beta^2) i}$ for $i > \frac{1.25}{\alpha}$. Thus the set of exponential components used for this example is complete and one can make ε as small as desired by increasing the number of poles in (1).

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8. R.E.A.C. Paley and N. Wiener, "Fourier Transforms in the Complex Domain," Am. Math. Soc. Colloquium Pubs., Vol XIX, Am. Math. Society, New York 1934.

It can be shown [7] that the roots of the following equation

$$\sum_{i=1}^{n+1} \sqrt{\Delta'_{ii}}\ \lambda^{(n-i+1)} = 0 \qquad (10)$$

yield the suboptimal exponents s_i, which are related to the optimum exponents s_i as $s_i' = s_i + \varepsilon\ell + \Theta(\varepsilon^2)$, where ℓ_i are the appropriate constants. Once the suboptimal exponents are found, the weights A_i can be obtained by a least square solution. The next section gives the solution procedure for the general case. The following result due to Gram is used to calculate ε. If $e_1,\ldots,e_n$ span an n-dimensional subspace S_n of an inner product space, then the least square approximant in S_n to a given $x \notin S_n$ is

$$g(t) = \frac{-1}{|G_n|} \begin{vmatrix} 0 & e_1 & \cdots & e_n \\ \hline < e_1, x> & & & \\ \vdots & & [G_n] & \\ <e_n, x> & & & \end{vmatrix}$$

where $[G_n] = [g_{ij} = <e_i, e_j>]$. In our application $e_i,\ldots..e_n$ are the exponential functions used to approximate x(t) $\{$i.e., $e_1 = \exp(s_1 t),\ldots.\}$. Hence,

$$\varepsilon = \frac{\sqrt{<\varepsilon r,\ x-g>}}{||r||} = \frac{\sqrt{\varepsilon <r,x>}}{||g||} = \frac{1}{||g||\ |G_n|^{\frac{1}{2}}} \begin{vmatrix} <x,x> & <e_1,x> & \cdots & <e_n,x> \\ \hline <e_1,x> & & & \\ \vdots & & [G_n] & \\ <e_n,x> & & & \end{vmatrix}^{\frac{1}{2}}$$

An alternative expression for ε can also be obtained as (11)

$$\varepsilon = \sqrt{\left[(||x||^2 / ||g||^2) - 1\right]} \qquad (12)$$

EXAMPLES

Four examples have been considered which deal with the approximation problem. They are approximations to the

(A) Square pulse $\left[\text{SQP}\right]$, where

$$\text{SQP} = \begin{cases} 1 & , \quad 0 \leq t \leq 1 \\ 0 & , \quad \text{elsewhere} \end{cases} \tag{13}$$

(B) Sine pulse $\left[\text{SINP}\right]$, where

$$\text{SINP} = \begin{cases} \text{Sin } \pi t & , \ 0 \leq t \leq 1 \\ 0 & , \ \text{elsewhere} \end{cases} \tag{14}$$

(C) Gaussian pulse $\left[\text{GP}\right]$, where

$$\text{GP} = \begin{cases} \exp\left\{-\left[\dfrac{t - 0.5}{0.25}\right]^2\right\} & ; \quad t \geq 0 \\ 0 & ; \quad t < 0 \end{cases} \tag{15}$$

and

(D) The function $\left[\text{TIP}\right]$

$$\text{TIP} = \begin{cases} (1+t^2)^{-1} & ; \quad t \geq 0 \\ 0 & ; \quad t < 0 \end{cases} \tag{16}$$

Figure 1 represents the approximation of SQP obtained by odd orders (n = 1, 3, 5, 7) of approximation. Figure 2 corresponds to the even orders (n = 2,4,6,8) of approximation. Figures 3 and 4 are the odd and even orders of approximation to the SINP. Similarly, Figures 5 and 6 correspond to the odd and even orders of approximation to the GP.

Figure 7 plots the integral squared error (ε) to the different orders of approximations for the SQP, SINP, and GP. As expected, it is easier to approximate a smooth function (e.g., GP) than a discontinuous

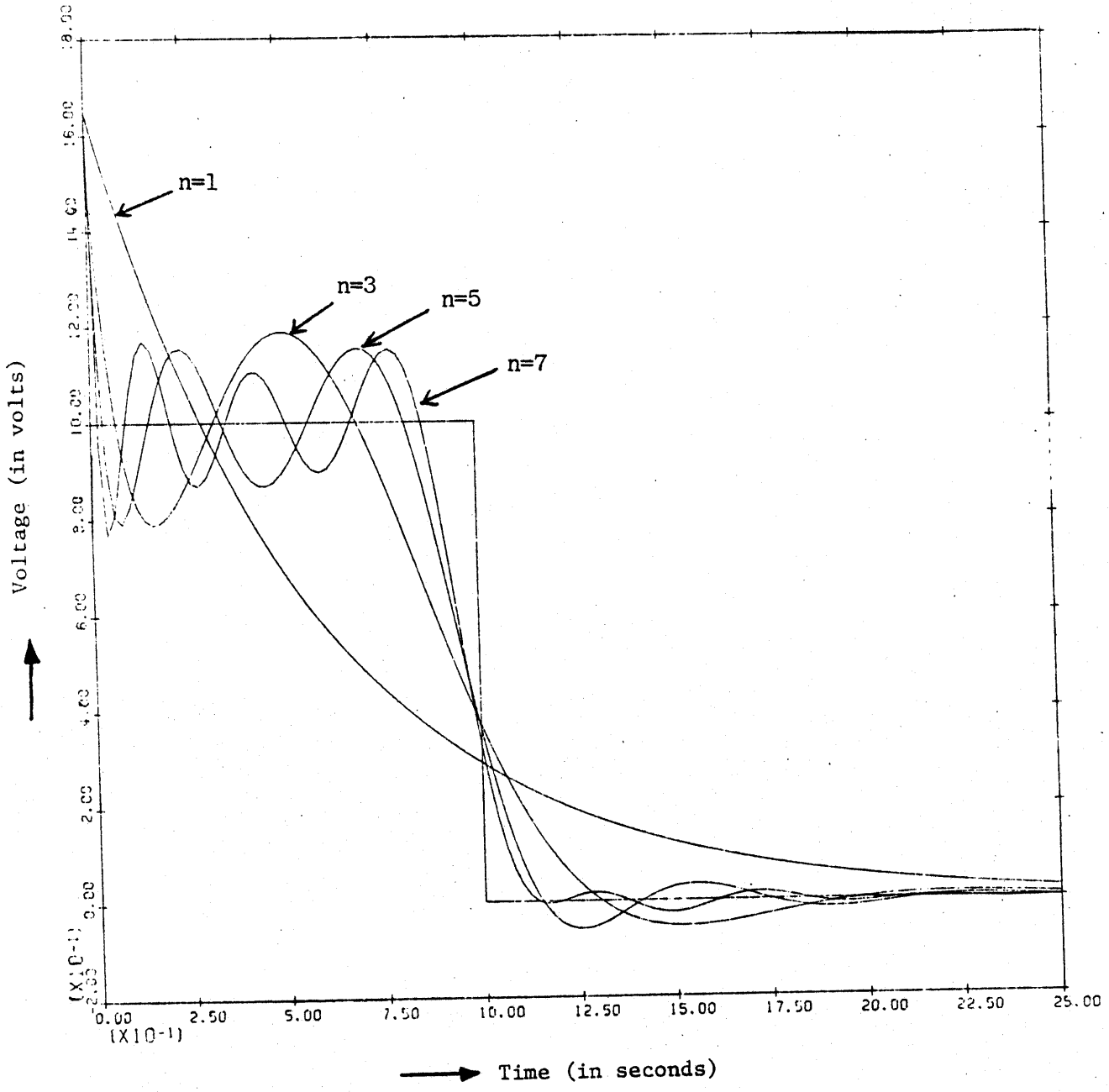

Fig. 1 Odd Order approximations of SQP

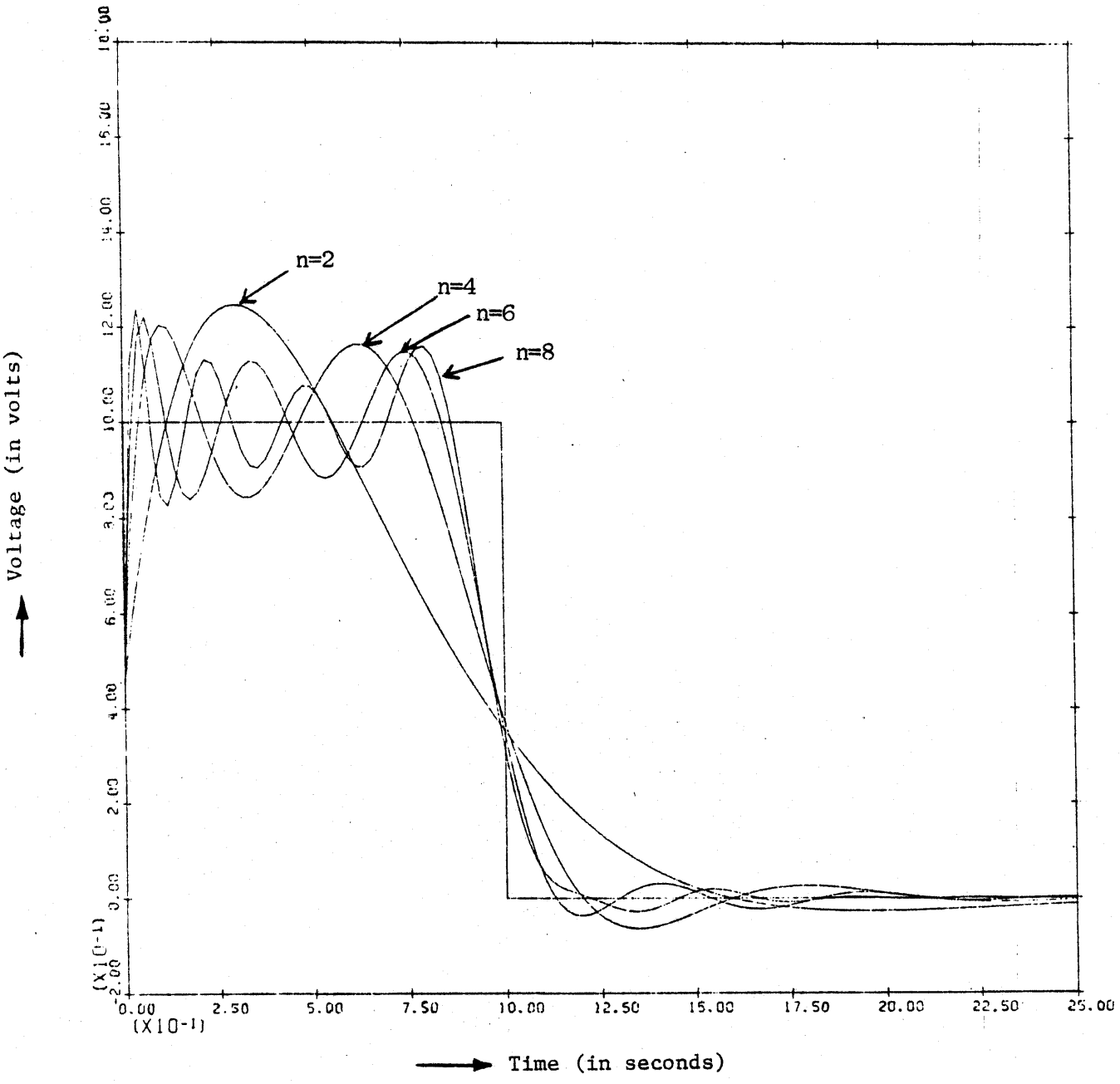

Fig. 2. Even Order approximations of SQP

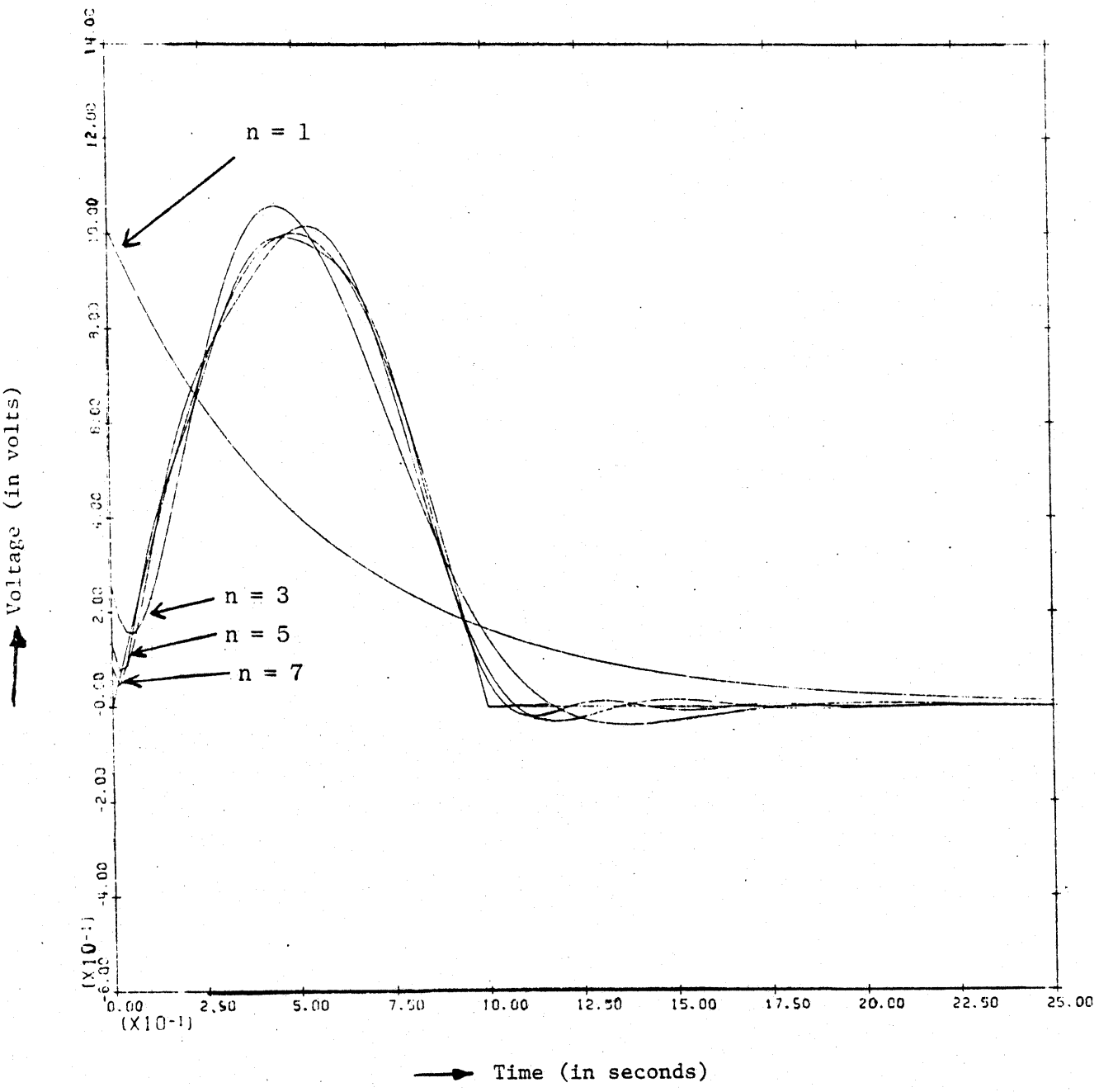

Fig. 3. Odd Order approximations of SINP

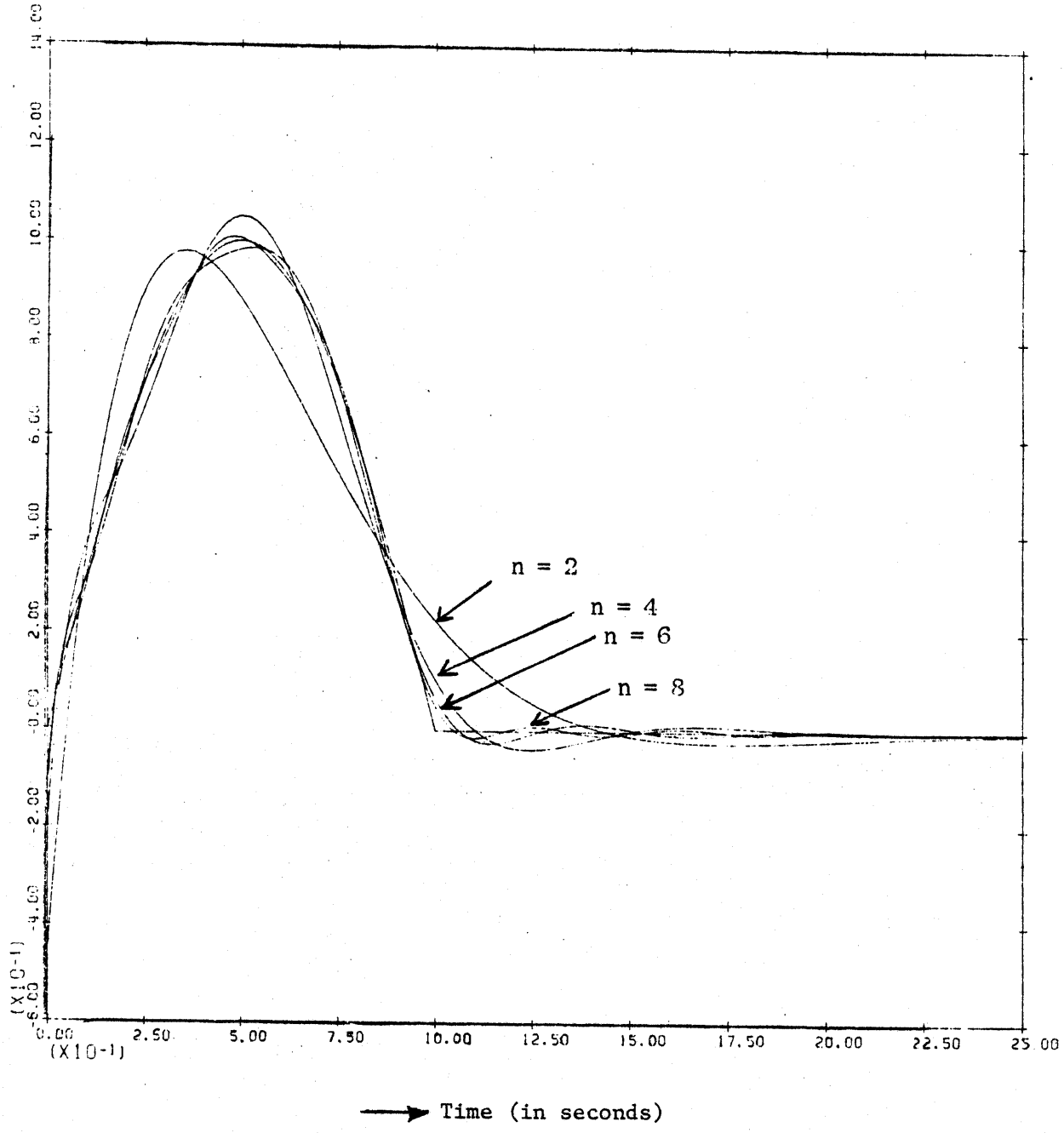

Fig. 4. Even Order approximations of SINP

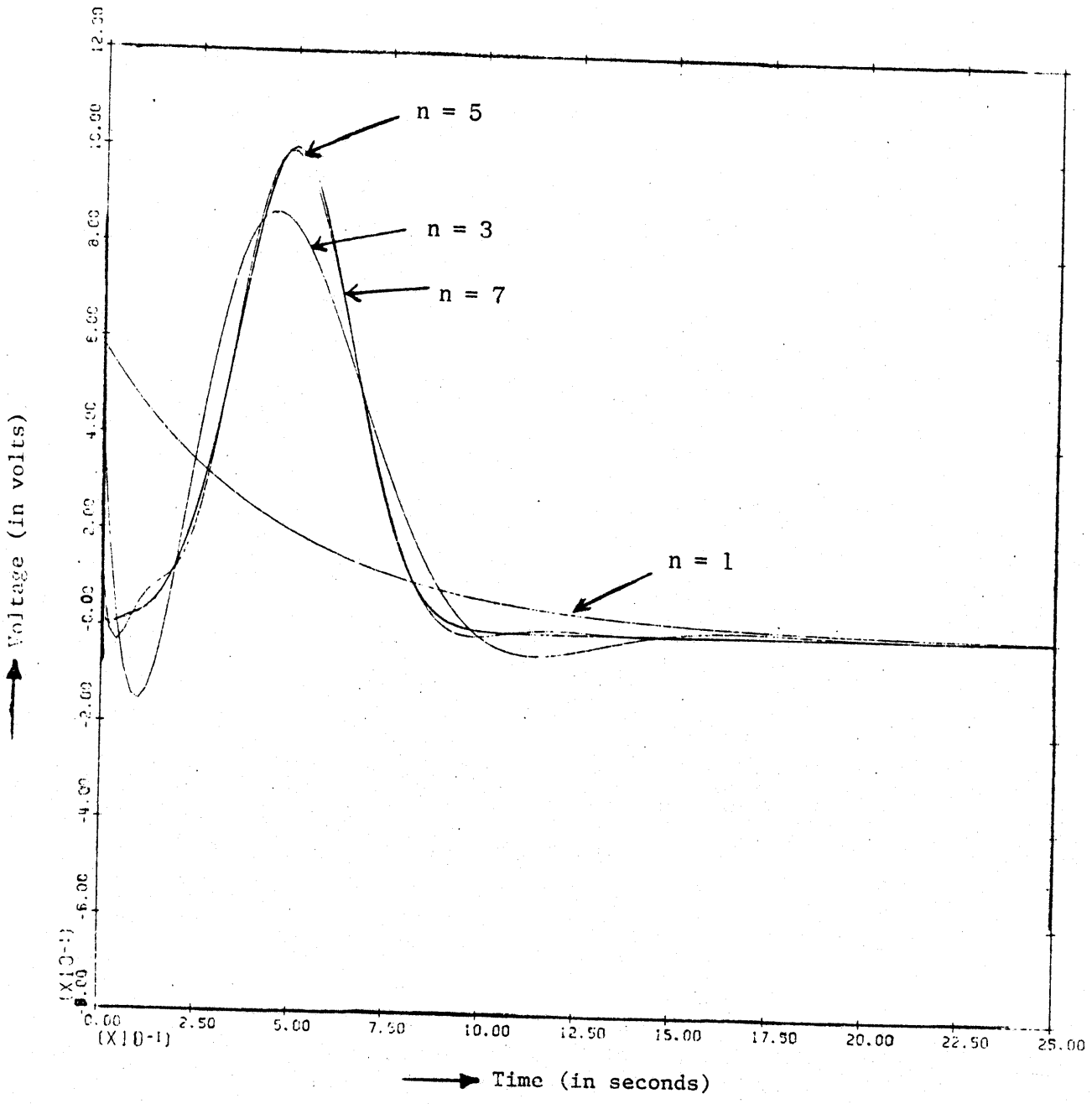

Fig. 5. Odd Order Approximations of GP

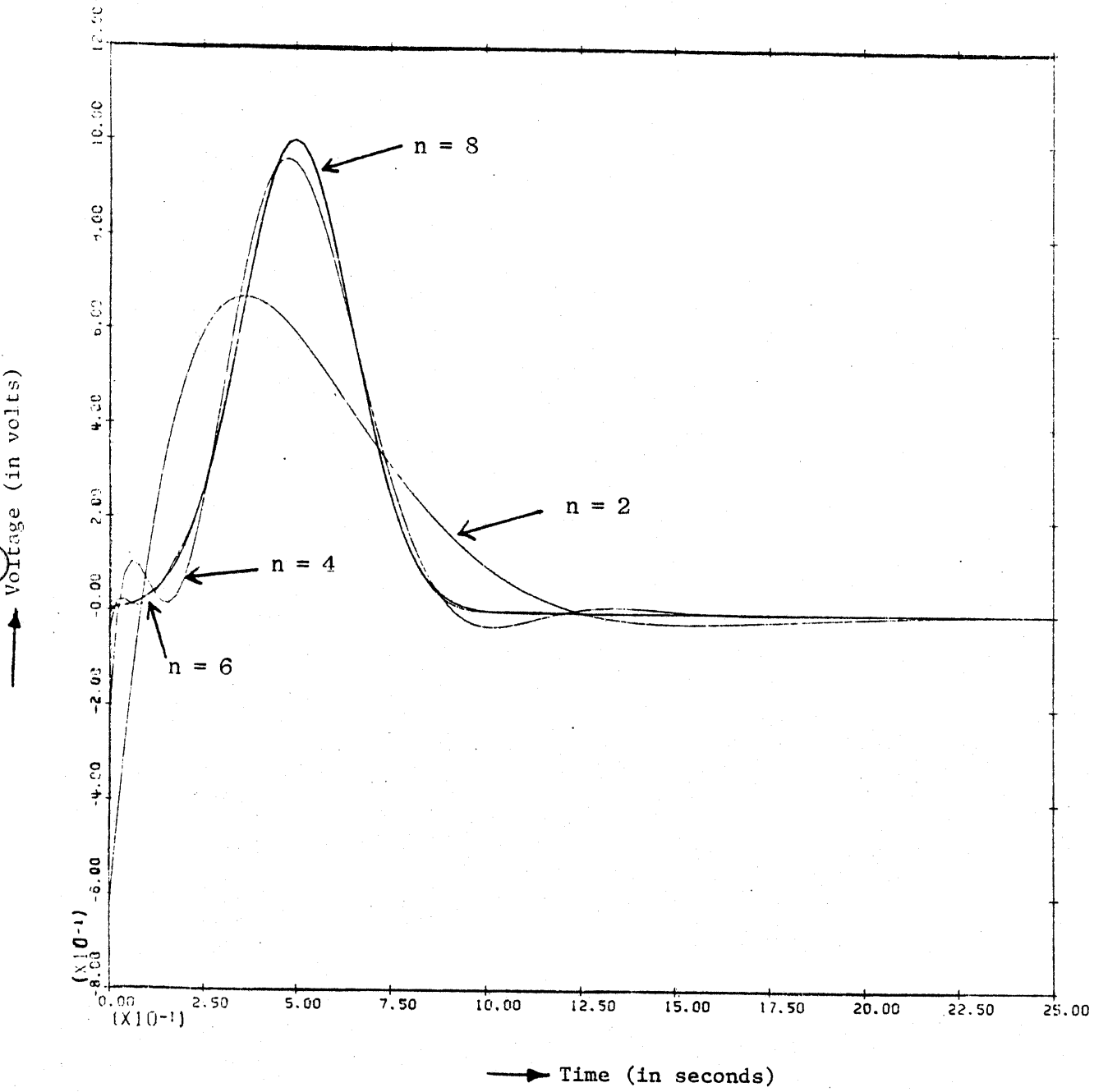

Fig. 6. Even Order approximations of GP

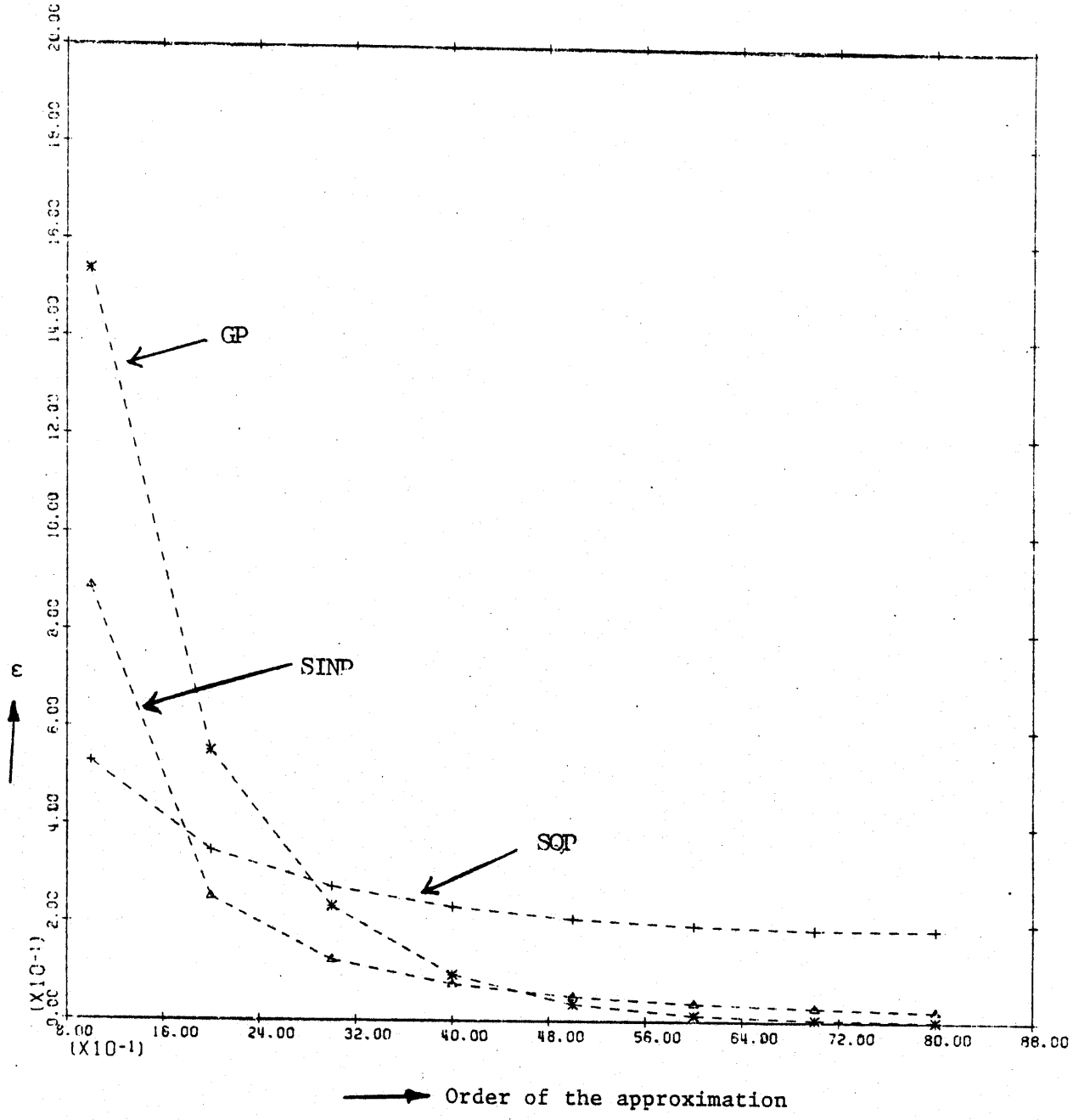

Fig. 7 Integral Squared Error (ε) for different pulse approximations

function (e.g., SQP). The plot also indicates that as the order of approximation is increased, ε decreases, thereby illustrating that this method converges numerically. Figures 8 and 9 represent odd and even approximations to TIP. Notice, largest absolute errors always occur at the origin.

This is a very significant point. It implies that when a signal starts from zero the sum of the weighting constants multiplying the damped exponentials must be zero. Hence, to begin with, the system is over determined.

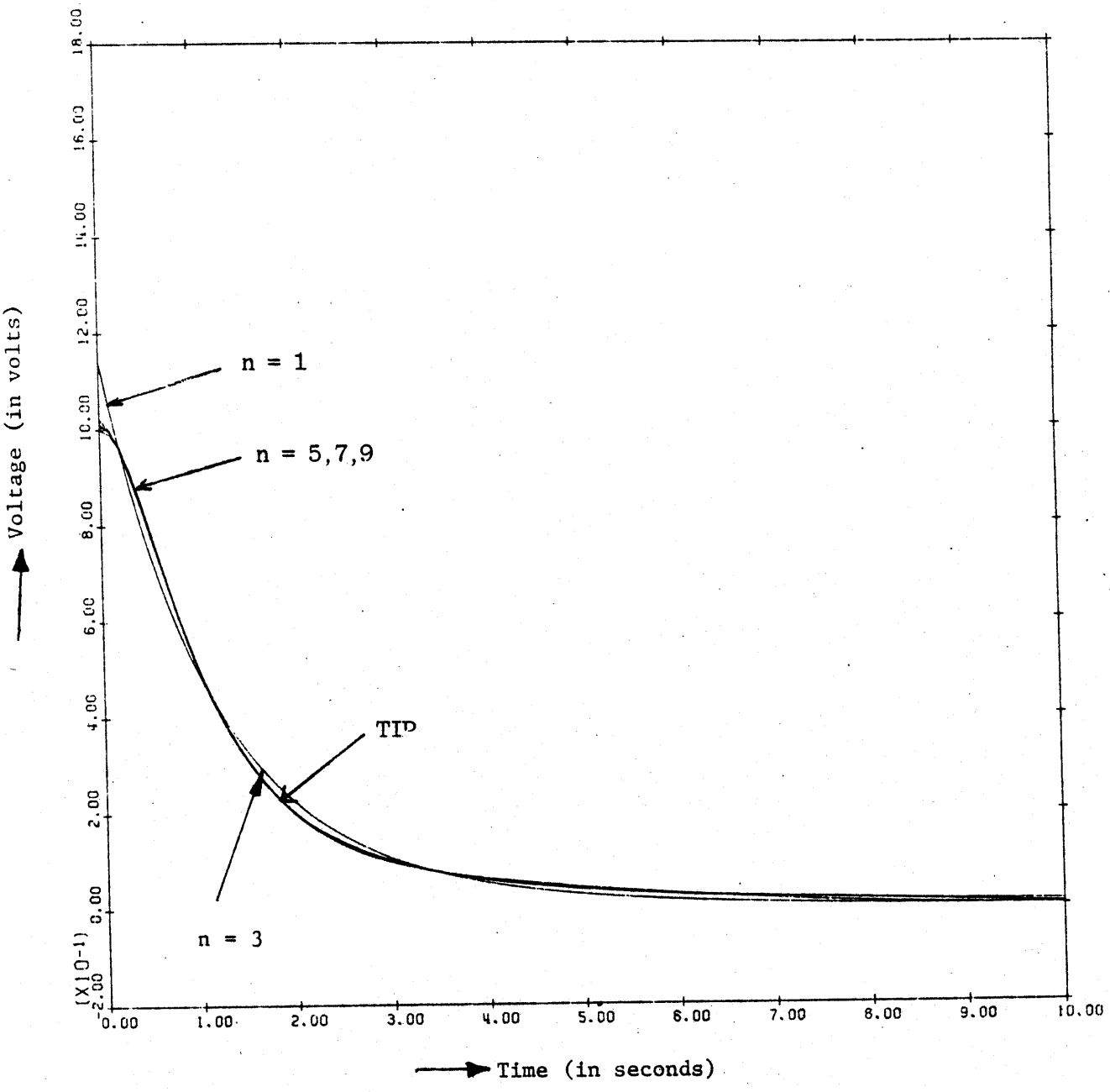

Fig. 8. True Impulse Response (TIP) and the approximated Odd Order Impulse Response

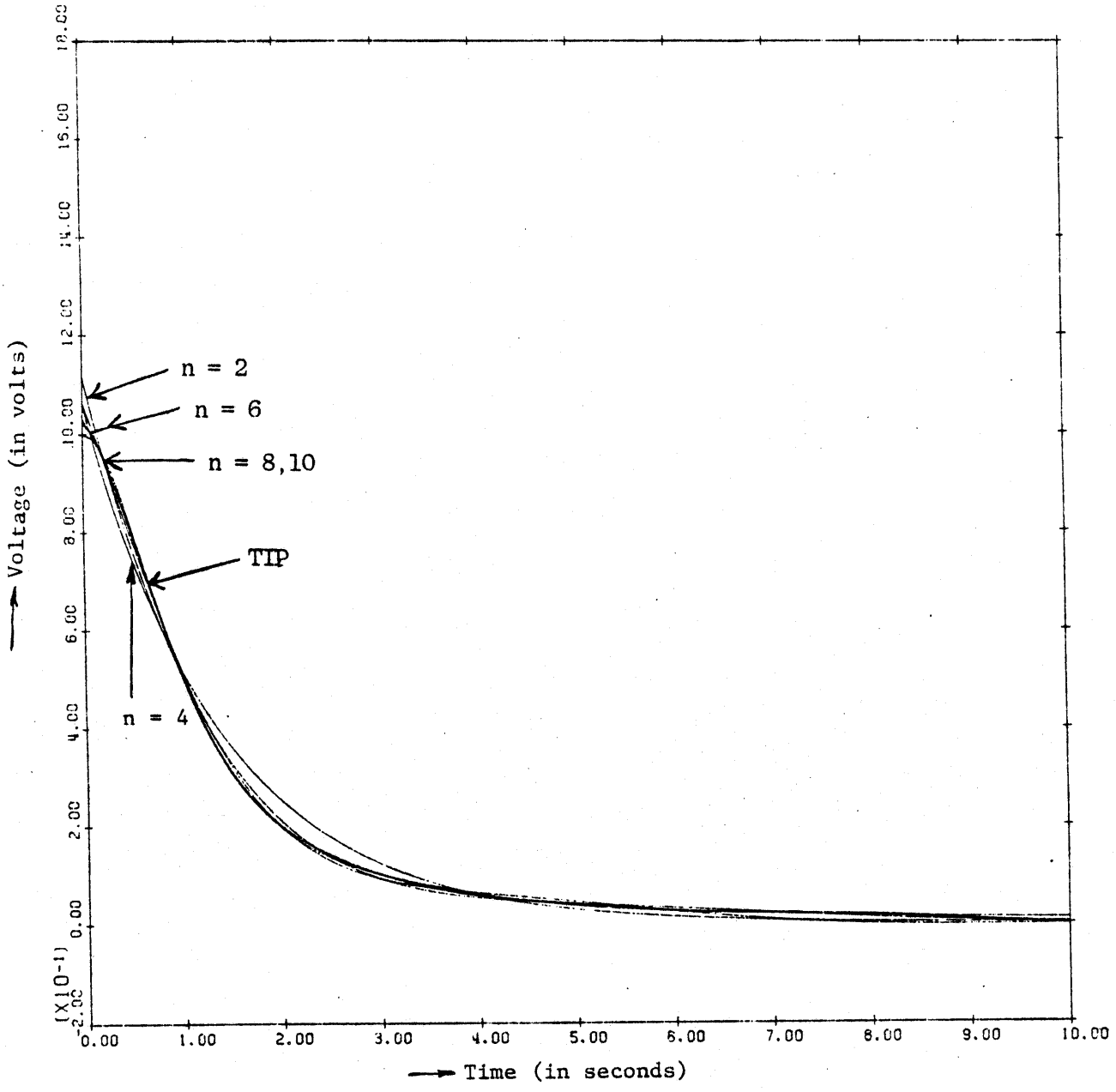

Fig. 9. True Impulse Response and the even order approximated Impulse Response

SECTION III

IDENTIFICATION OF SYSTEM POLES AND RESIDUES: ARBITRARY INPUT WITH ZERO INITIAL STATE

The complex exponential approximation of square integrable signals over $[0,\infty)$ lends itself in a very natural way to the system identification problem for causal systems. In practice, an infinite observation interval is not available. (The sample should be truncated in a way to include most of the energy.) Secondly, for identification of a system under normal operating conditions, the available data records are usually for some arbitrary input. The method presented here incorporates the two practical situations mentioned above. The problem then is to obtain the system poles and the residues at the poles when the output y (t) of the system is known for a known input x(t). The impulse response h (t) of the system is assumed to be of the form described by (1) and (2). Hence the transfer function H (s) is obtained by taking the Laplace Transform of (1) as

$$H(s) = \sum_{i=1}^{q} \sum_{j=1}^{m_i} A_{ij} \frac{(j-i)!}{(s-s_i)^j}$$

For zero initial state, the output y (t) of the system is obtained as

$$y(t) = \int_0^t x(\tau)\, h(t-\tau)\, d\tau$$

$$= \sum_{i=1}^{q} \sum_{j=1}^{m_i} A_{ij} \int_0^t (t-\tau)^{(j-1)} \cdot e^{s_i (t-\tau)} x(\tau)\, d\tau \qquad (17)$$

The Laplace transform of y(t) is then obtained as

$$Y(s) = Y_1(s) = X(s)\ H(s) = \sum_{i=1}^{q} \sum_{j=1}^{m_i} A_{ij} \frac{(j-i)!}{(s-s_i)^j} X(s) \qquad (18)$$

If the output time function is integrated n times, then the

$$y_{n+1}(t) = \int_0^t y_n(\tau)\, d\tau, \text{ where } y_1(t) = y(t) \qquad (19)$$

Laplace Transform of (19) is expressed as

$$Y_{n+1}(s) = \sum_{i=1}^{q} \sum_{j=i}^{m_i} A_{ij} \frac{(j-i)!}{(s-s_i)^j} \frac{X(s)}{s^n} = \frac{Y_1(s)}{s^n}$$

Next consider two functions, p(t) and d(t), which are defined on a common interval. The two functions are combined via a scalar parameter λ to produce

$$f(t,\lambda) \triangleq \lambda\, p(t) + d(t) \qquad (20)$$

The quantity $f(t, \lambda)$ is defined to be a pencil of functions p(t) and d(t) parameterized by λ [6]. It is a linear combination of functions d(t) and p(t) through the parameter λ. The pencil of functions contains very important characteristics from a system identification point of view, when the functions d(t), p(t) and the parameter λ are appropriately selected.

The set of 2n pencil functions defined by

$$\left[\left\{y_1(t) - \lambda y_2(t)\right\}, \left\{y_2(t) - \lambda y_3(t)\right\}, \ldots, \left\{y_n(t) - \lambda y_{n+1}(t)\right\},\right.$$

$$\left. x_2(t), x_3(t), \ldots, x_{n+1}(t)\right] \qquad (21)$$

is considered. Since in this case it is simpler to deal with the frequency domain quantities, attention is focused on the set of pencil functions defined by

$$\left[\left\{Y_1(s) - \lambda\, Y_2(s)\right\}, \left\{Y_2(s) - \lambda\, Y_3(s)\right\}, \ldots, \left\{Y_n(s) - \lambda\, Y_{n+1}(s)\right\},\right.$$

$$\left. X_2(s), X_3(s), \ldots X_{n+1}(s)\right] \tag{22}$$

This set of pencil functions displays some interesting properties if they are checked for linear independence. Checking for linear independence of the set, we write

$$\sum_{k=1}^{n} \left[a_k \left\{Y_k(s) - \lambda\, Y_{k+1}(s)\right\} + b_k X_{k+1}(s)\right] = 0 \tag{23}$$

Substitution of (18) into (23) yields.

$$\sum_{k=1}^{n} s^{n-k} \left[a_k (s-\lambda) H(s) + b_k\right] = 0 \tag{24}$$

where

$$H(s) = \frac{N(s)}{D(s)} = \frac{\prod_{\ell=1}^{q} (s - s_\ell)^{m_\ell} \sum_{i=1}^{q} \sum_{j=1}^{m_i} A_{ij} \frac{(j-i)!}{(s - s_i)^j}}{\prod_{\ell=1}^{q} (s - s_\ell)^{m_\ell}}$$

Clearly, (24) is independent of the choice of $X_1(s)$. Hence, the input function $X_1(s)$ does not affect the linear independence of the set of pencil functions in (24) unless $X_1(s) = 0$. Therefore, any conclusions drawn on the properties of the pencil of functions of

(22) do not depend on the input functions since only nonzero input functions are employed in the analysis. D(s) is a polynomial of order n in s. Its roots are the system poles. N(s) is a polynomial in s having a highest possible order of (n-1). Equation (24) can also be rewritten as

$$\sum_{k=1}^{n} s^{n-k} \left[a_k (s-\lambda)\, N(s) + b_k \prod_{i=1}^{q} (s-\lambda_i)^{m_i} \right] = 0 \qquad (25)$$

It is assumed that any common roots to N(s) and D(s) have been cancelled before substitution into (25). Equation (25) represents a polynomial equation in s of the order (2n-1). Since 1, s, s^2,... s^{2n-1} span a 2n dimensional subspace, this set is linearly independent and hence the set of 2n coefficients (a_k, b_k) must be identically zero if (25) is to be satisfied for all values of s. The above holds as long as $\lambda \neq \lambda_i$. For $\lambda = \lambda_i$, it turns out that (25) has a root $s_i = \lambda_i$ which can be factored out. The polynomial equation now reduces to an order of (2n-2). The set 1, s, s^2, ...s^{2n-2} spans a $2n-1$ dimensional subspace. Nevertheless, there remains $2n$ set of coefficients (a_k, b_k). It is now possible for equation (25) to be satisfied with at least one nonzero coefficient. This implies linear dependence of the set (22), when $\lambda = \lambda_i$. So if λ is a system pole then the Gram determinant of the set of the functions will be zero. The Gram determinant of the given set (21) is defined by

$$\left[G_{2N}\right] = \begin{bmatrix} \langle y_1 - \lambda y_2,\ y_1 - \lambda y_2 \rangle & \langle y_1 - \lambda y_2,\ y_2 - \lambda y_3 \rangle & \cdots\cdots & \langle y_1 - \lambda y_2,\ x_{n+1} \rangle \\ \vdots & \vdots & & \vdots \\ \langle x_{n+1},\ y_1 - \lambda y_2 \rangle & \langle x_{n+1},\ y_2 - \lambda y_3 \rangle & \cdots\cdots & \langle x_{n+1},\ x_{n+1} \rangle \end{bmatrix}$$

Thus, $\det\left[G_{2N}\right] = 0$, if λ is a system pole. After some tedious algebraic manipulations, it can be shown that

$$\det\left[G_{2N}\right] = \left\{ \sum_{i=1}^{n+1} \sqrt{\Delta_{ii}}\ \lambda^{n-i+1} \right\} \left\{ \sum_{j=1}^{n+1} \sqrt{\Delta_{jj}}\ \lambda^{n-j+1} \right\}$$

where Δ_{ii} is the ith diagonal cofactor in the Gramian $\left[G_{2n+1}\right]$ and only the first n+1 cofactors are considered. The Gram matrix $\left[G_{2n+1}\right]$ is defined as

$$\left[G_{2n+1}\right] = \begin{bmatrix} \langle y_1, y_1\rangle & \cdots & \langle y_1, y_{n+1}\rangle & \langle y_1, x_2\rangle & \cdots & \langle y_1, x_{n+1}\rangle \\ \vdots & & \vdots & \vdots & & \vdots \\ \langle y_{n+1}, y_1\rangle & \cdots & \langle y_{n+1}, y_{n+1}\rangle & \langle y_{n+1}, x_2\rangle & \cdots & \langle y_{n+1}, x_{n+1}\rangle \\ \langle x_2, y_1\rangle & \cdots & \langle x_2, y_{n+1}\rangle & \langle x_2, x_2\rangle & \cdots & \langle x_2, x_{n+1}\rangle \\ \vdots & & \vdots & \vdots & & \vdots \\ \langle x_{n+1}, y_1\rangle & \cdots & \langle x_{n+1}, y_{n+1}\rangle & \langle x_{n+1}, x_2\rangle & \cdots & \langle x_{n+1}, x_{n+1}\rangle \end{bmatrix} \quad (26)$$

The polynomial equation, whose roots are the poles locations, is then given by

$$\sum_{i=1}^{n+1} \sqrt{\Delta_{ii}}\ \lambda^{n-i+1} = 0 \quad (27)$$

Instead of observing the magnitude of the square root of the Gram determinant another test can be used. The new test criterion uses the ratio of the geometric and arithmetic means of the eigenvalues

$$D_M = \frac{\left[\prod_{i=1}^{M} p_i\right]^{1/M}}{\frac{1}{M}\left[\sum_{i=1}^{M} p_i\right]}$$

D_M is defined as the ill conditioning measure of the Gram matrix. For an Mth order matrix p_i are the eigenvalues of $\left[G_{2n+1}\right]$. However, this test can be applied without any spectral computation. This is because the product of the eigenvalues equals the determinant of the matrix and the sum of the eigenvalues equals the trace. Since the geometric mean must be less than or equal to the arithmetic mean, the ill conditioning measure D_M must always lie between zero and one. It can equal zero only if $\left|G_{2n+1}\right| = 0$, i.e. only if $\left[G_{2n+1}\right]$ is singular. A value of D_M equal to unity results only if $\left[G_{2n+1}\right]$ is proportional to an indentity matrix. Since the identity matrix is trivial to invert, a value of D_M equal to unity implies that $\left[G_{2n+1}\right]$ is perfectly conditioned. It follows that D_M leads to a more quantitative explanation for the following qualitative statements:

1) More accuracy is needed to analyze real data than synthetic data.
2) Raising the sampling rate increases the amount of computational accuracy needed (since the higher frequency energy of real data generally decreases as frequency increases). Hence, there exists an optimum sampling rate for any waveform.
3) Proper prefiltering can decrease the amount of computational accuracy needed.

The output y(t) can now be expressed as

$$y(t) = \sum_{i=1}^{q} \sum_{j=1}^{m_i} A_{ij} P_{ij}$$

$$\triangleq \sum_{i=1}^{q} \sum_{j=1}^{m_i} A_{ij} e^{s_i t} \int_0^t (t-\tau)^{j-1} e^{-s_i \tau} x(\tau) d\tau \qquad (28)$$

Once the P_{ij}'s are obtained, the unknown constants A_{ij} can be obtained from a least squares procedure.

The integrated squared error ε can be found as described earlier.

EXAMPLES

Several examples have been considered to illustrate this technique.

A. Input: $\sin(t)$

Output: $5\left[\exp(-t) + \sin(t) - \cos(t)\right]$

Data: 101 uniformly spaced samples of a record of 2 seconds.

The true transfer function is

$$H(s) = \frac{10}{s+1}$$

A first order approximation was chosen and the result is

$$H'(s) = \frac{10.0016396}{s + 1.0001723}$$

B. Input: $\exp(-2t)\cos(t)$

Output: $0.5\left[\exp(-t)-\exp(-3t)\right]$

Data: 201 uniformly spaced samples of a record of 3 seconds.

The true transfer function is

$$H(s) = \frac{(s-2-j)\ (s-2+j)}{(s+1)\ (s+2)\ (s+3)}$$

A third order approximation indicates

$$H'(s) = \frac{.83319(s-2.00099 - .998619j)\ (s-2.00099 + .998619j)}{(s+.999979)\ (s+1.99997)\ (s+2.99949)}$$

The gain factor in this case deviates from unity because of the truncation error.

C. Input: ramp (t)

Output: $6\left[-1 + t + \exp(-t)\cos(t)\right]$

Data: 201 uniformly spaced samples of a record of 4π seconds.

The true transfer function is

$$H(s) = \frac{12}{s^2 + 2s + 2}$$

A second order approximation yields

$$H'(s) = \frac{-.0058987\ s + 12.0158425}{s^2 + 2.0021149\ s + 2.0025363}$$

The contribution of the additional zero is negligible for any practical purposes.

Next the technique is applied to the synthesis of poles and zero for noise contaminated data. The empirical signal is $2e^{-t}\left[\cos 2t + 2 \sin 2t\right]$. The noise is introduced as a form of multiplicative noise. A random number is generated; and if the number is even the value of the empirical signal is multiplied by 1.05 for 5% noise, and if an odd random number is generated the multiplicative factor becomes 0.95.

The results of the poles and residues are first given for Prony's method and then for the present technique.

A. Prony's method

1. No Noise

Poles: $-1 \pm j2$

Residues: $1 \mp j2$

2. 5% Multiplicative Noise (-26dB)

case a. Poles: $-1.298717 \pm j\ 1.676503$

Residues: $0.95 \pm j\ 0.671413$

case b. Poles: $-.252421;\quad -1.207952$

Residues: $-.01973812;\quad 1.919738$

The two different cases are due to two sets of random noise parameters

B. Present Method

1. No Noise

Poles: $-1.001384 \pm j\ 2.000112$

Residues: $1.000154 \mp j\ 1.999721$

2. With 5% Multiplicative Noise (-26dB)

case a. Poles: $-1.016796 \pm j\ 2.001249$

Residues: $1.020853 \mp j\ 2.037931$

case b. Poles: $-.9977843 \mp j\ 1.980839$

Residues: $1.043612 \pm j\ 1.941599$

3. With 20% Multiplicative Noise (-14dB)

case a. Poles: $-.965332 \pm j\ 2.011705$

Residues: $.949282 \mp j\ 2.024689$

case b. Poles: $-1.014091 \pm j\ 2.000726$

Residues: $1.158494 \mp j\ 1.954903$

Even with 20% noise the maximum error in the location of the poles is 3.5% in the real part and 2% in the imaginary part, whereas for the residues they are 16% in the real part and 5% in the imaginary part. The Prony's method gave erratic results even with 1% multiplicative noise.

The present technique is next applied to the identification of poles and residues in a synthetic signal with additive noise. The signal is the same as before. The noise is uniformly distributed between $\pm .04$ so that $\sigma \simeq .025$; 300 equispaced sample points were chosen of a record of 8 seconds. The results obtained are

Poles: $-.86644 \pm j\ 1.8323$

Residues: $1.0065 \mp j\ 1.7617$

As a final example, consider the transient response of a 10m long and 1m diameter conducting pipe tested at the ATHAMAS-I EMP simulator (or popularly known as the HPD). The time domain responses of the pipe were measured at different points in space. One such response [9] is shown in Figure 10. The particulars of the experiment are as follows:

Data Nos.: PH 00202

Shot: 4316

Sensor No.: 0026

Meas No.: J1

Orientation: 00

Test Item pos.: +00 +02 +30

Configuration: 001

Environmental level: 111

The technique mentioned in this paper was applied on the truncated portion of the waveform (from 0 to 500 nanoseconds). The poles and their residues obtained for different order approximations are shown below.

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9. J. S. Yu, C-L J. Chen and J. P. Castillo, "Responses of a Conducting Pipe," ATHAMAS Memo - 14, April 14, 1977.

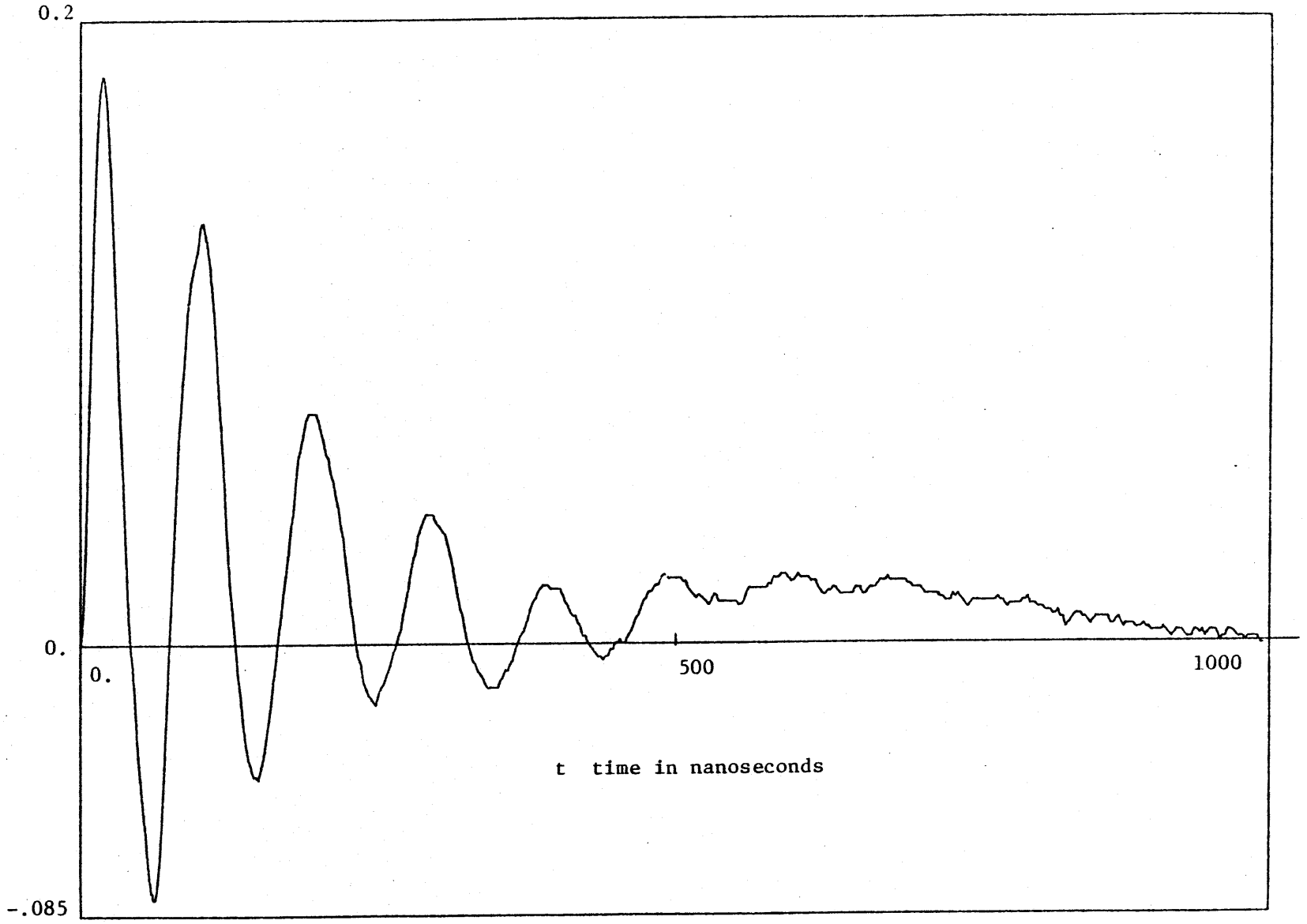

Fig. 10. Transient Response of a Conducting Pipe Measured at the ATHAMAS-I EMP Simulator

For n = 5, the poles and the residues are

Poles (in rad/sec) x 10^7	Residues
$-1.339 + j\ 0.$	$-.5396 - j\ 0.$
$-1.080 \mp j\ 1.158$	$.1430 \pm j\ .2811$
$-1.750 \mp j\ 5.480$	$.1401 \pm j\ .0798$

$D_5 = .0399$

For n = 6, the poles and the residues are

Poles (in rad/sec) x 10^7	Residues
$-1.466 \pm j\ .5716$	$-.2981 \mp j\ .6743$
$-.9696 \mp j\ 1.660$	$.2189 \mp j\ .0218$
$-.7370 \mp j\ 6.308$	$.0895 \pm j\ .0364$

$D_6 = .0130$

For n =7, the poles and the residues are

Poles (in rad/sec) x 10^7	Residues
-1.588 + j 0.	-1.288 - j0.
- .6428 $\mp$ j 1.272	- .4220 $\mp$ j .0274
-1.071 $\mp$ j 1.262	.9802 $\pm$ j .3593
- .7012 $\mp$ j 6.321	.0918 $\pm$ j .0333

$D_7 = .0060$

Unfortunately, it was not possible to go beyond seventh order because of limitations in the computer program. It is quite clear that the pole which has converged is

$$s_d = (- .7012 \mp j\ 6.321) \times 10^7 \text{ rad/sec.}$$

The dominant pole of the same waveform was obtained by Cordaro* using an iterative Prony method as

$$s_c = (- .603 \mp j\ 6.75) \times 10^7 \text{ rad/sec.}$$

There seems to be close agreement between the results.

The total time taken by the present method to analyze all of the approximations up to seventh order on an IBM 370/155 computer was appromimately 1 minute.

Also, we point out that the convergent pole is the dominant pole of the desired waveform as contrasted with the measured waveform. This is because the measured response is the integral of the desired waveform as shown in Figure 10. Hence, the residues associated with the measured waveform are modified by factors involving the respective poles.

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* Private Communication with Dr. T. Cordaro

That is, if

$$x(t) = \sum_i a_i e^{\lambda_i t}$$

then

$$\int_{\infty}^{t} x(\tau)\, d\tau = \sum_i \frac{a_i}{\lambda_i} e^{\lambda_i t}$$

s_d emerges as the dominant pole of the desired waveform when the residues of the measured waveform are multiplied by the corresponding pole values.

Also, it is seen that the value of D_M decreases as the order increases. D_M equal to zero represents a singular matrix and hence an overdetermined system. For the seventh order $D_7 = 0.006$, which represents a near singular matrix.

It is expected that proper prefiltering (i.e. use of first order filters instead of pure integrators) and utilization of the knowledge of the input waveform to the system may significantly increase convergence of the poles. Then the values of D_M would decrease at a faster rate than is observed presently.

SECTION IV

IDENTIFICATION OF SYSTEM POLES AND RESIDUES: ARBITRARY INPUT WITH NONZERO INITIAL STATE

When the initial state is not zero, the output can be expressed as

$$y(t) = \sum_{i=1}^{q} \left[\sum_{j=1}^{m_i} A_{ij} \int_0^t (t-\tau)^{j-1} e^{-s_i \tau} x(\tau) d\tau + a_i \right] e^{s_i t}$$

where a_i are related linearly to the initial state. For this case it can be shown that the pencil set

$$\left\{ y_1 - \lambda y_2, \ldots, y_n - \lambda y_{n+1}; x_2 \ldots x_{n+1}; t, \ldots t^{n-1} \right\}$$

is linearly dependent if and only if λ is one of the system poles. The development is similar to that of the previous section except that the Gram matrix is now (3n+1) dimensional as defined by the functions

$$\left\{ y_1, \ldots, y_{n+1}; x_2, \ldots, x_{n+1}; 1, t, \ldots t^{n-1} \right\}$$

SECTION V

DISCUSSION

The suboptimal exponents determined nonitatively are $s_i' = s_i + \varepsilon\ell_i$. Thus as the order of approximation n is increased, not only should the error decrease but the suboptimal approximation will approach the optimum. Also a formula has been developed for the direct computation of the error. The major disadvantage of the technique is that the Gram matrix may be ill-conditioned in some problems, especially when the order of the system n becomes large.

However, when the data are noisy the integrators in the above system can be replaced by FOF (first order filters) to enhance accuracy in the estimation procedure. In addition, realization of first order filters is easier to obtain than is the realization of integrators when a hybrid implementation is desired. Presently, work is being carried out to study the different aspects of this problem.

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