Mathematics Notes

Note 52

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Errors in the Fourier Transformation of a Double Exponential Due to Time Truncation

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Abstract

The effects of time truncation on the Fourier transform of a typical double exponential ($e^{-1.5x10^6 t} - e^{-2.6x10^8 t}$) are investigated. Time truncation is a problem when the Fourier transform is taken of a function which has not yet decayed to zero, i.e., time stops at a finite limit and does not go to infinity as the Fourier transform requires. The procedure used was to calculate the analytical Fourier transform of a time truncated double exponential both with and without selected taperings. The difference in percent of the magnitude of the ideal transform and the time truncated transform was calculated for various frequencies. The percent error was then contour plotted with independent axes of frequency and truncation time.

Contents

Introduction

Time truncation occurs frequently in EMP transient simulator measurement systems since the transient has a relatively high rise rate and a relatively low decay rate. If one goes to a high sweep speed to look at the early time data, time truncation occurs because the whole waveform cannot be displayed. If one uses a low sweep speed to look at all the data, the early time resolution is lost. To give some insight into the effects of time truncation on the Fourier transforms of such data, a particular double exponential was selected and the effects of truncation on the analytic Fourier transformation was calculated. The effects of two tapering techniques were compared with those of outright truncation. Time domain waveforms with negligible DC (i.e., zero frequency) content (e.g., damped sines) were not investigated.

I. Theory

1. Introduction.

The Fourier transform of a function f is

$$F(\omega) \triangleq \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt \quad .$$

Assuming $f(t) = 0$ for all $t < 0$, then

$$F(\omega) = \int_0^{\infty} f(t)\, e^{-i\omega t}\, dt =$$

$$= F_\omega(t)\Big|_{t=0}^{\infty} \qquad (1)$$

where

$$F_\omega(t) \triangleq \int f(t)\, e^{-i\omega t}\, dt \qquad (2) \quad .$$

2. Truncation without tapering.

When there is truncation then f(t) is known only up to some time T_{max}, but may be nonzero (but unknown) for t outside the interval $[0,T_{max}]$. Let

$$G_\omega(t) \triangleq \int_0^t f(\tau)\, e^{-i\omega\tau}\, d\tau = F_\omega(\tau)\Big|_{\tau=0}^{t} \qquad (3) \quad .$$

By linearity of the Fourier integral

$$F(\omega) = \int_0^{T_{max}} f(t)\, e^{-i\omega t}\, dt + \int_{T_{max}}^{\infty} f(t)\, e^{-i\omega t}\, dt =$$

$$= G_\omega(T_{max}) + F_\omega(t)\Big|_{t=T_{max}}^{\infty} \quad .$$

Let

$$E_\omega(t) \triangleq F_\omega(\tau)\Big|_{\tau=t}^{\infty} = F_\omega(\infty) - F_\omega(t) \qquad (4) \quad .$$

Then we have

$$F(\omega) = G_\omega(T_{max}) + E_\omega(T_{max}) \tag{5}$$

$F(\omega)$ is the transform of the function f over all time and $G_\omega(T_{max})$ is the approximation of the transform which we calculate because of the fact that we have data only up to T_{max}. $E_\omega(T_{max})$ is the error in the calculated approximation due to truncation.

Consider the case in which f(t) is the double exponential

$$f(t) = \begin{cases} 0 & \forall \;\; t < 0 \\ e^{-\alpha t} - e^{-\beta t} & \forall \;\; t \geq 0 \end{cases} \tag{6}$$

In this case equation (1) yields

$$\begin{aligned} F(\omega) &= \int_0^\infty (e^{-\alpha t} - e^{-\beta t})\, e^{-i\omega t}\, dt = \\ &= \int_0^\infty e^{-(\alpha+i\omega)t}\, dt - \int_0^\infty e^{-(\beta+i\omega)t}\, dt = \\ &= \left[\frac{-1}{\alpha+i\omega} e^{-(\alpha+i\omega)t} + \frac{1}{\beta+i\omega} e^{-(\beta+i\omega)t} \right]_{t=0}^{\infty} = \\ &= \frac{1}{\alpha+i\omega} - \frac{1}{\beta+i\omega} \end{aligned} \tag{7}$$

equation (3) yields

$$\begin{aligned} G_\omega(T_{max}) &= \int_0^{T_{max}} (e^{-\alpha t} - e^{-\beta t})\, e^{-i\omega t}\, dt = \\ &= \left[\frac{-1}{\alpha+i\omega} e^{-(\alpha+i\omega)t} + \frac{1}{\beta+i\omega} e^{-(\beta+i\omega)t} \right]_{t=0}^{T_{max}} = \\ &= \frac{-1}{\alpha+i\omega} [e^{-(\alpha+i\omega)T_{max}} - 1] + \frac{1}{\beta+i\omega} [e^{-(\beta+i\omega)T_{max}} - 1] = \\ &= \frac{1 - e^{-(\alpha+i\omega)T_{max}}}{\alpha+i\omega} - \frac{1 - e^{-(\beta+i\omega)T_{max}}}{\beta+i\omega} \end{aligned} \tag{8}$$

and either equation (4) or equation (5) yields

$$E_{\omega}(T_{max}) = \frac{e^{-(\alpha+i\omega)T_{max}}}{\alpha+i\omega} - \frac{e^{-(\beta+i\omega)T_{max}}}{\beta+i\omega} \qquad (9) \quad .$$

In this case then

$$F(0) = \frac{1}{\alpha} - \frac{1}{\beta}$$

and

$$E_0(T_{max}) = \frac{e^{-\alpha T_{max}}}{\alpha} - \frac{e^{-\beta T_{max}}}{\beta} \quad .$$

Since normally $10\alpha < \beta$, therefore the α term usually dominates the right hand sides of the last two equations. That is,

$$F(0) \doteq \frac{1}{\alpha}$$

and

$$E_0(T_{max}) \doteq \frac{e^{-\alpha T_{max}}}{\alpha} \quad .$$

Consequently a good approximation of the *relative* error in the truncated estimate $G_0(T_{max})$ of $F(0)$ at DC is given by

$$\frac{E_0(T_{max})}{F(0)} \doteq \left(\frac{e^{-\alpha T_{max}}}{\alpha}\right) \Big/ \frac{1}{\alpha} =$$

$$= e^{-\alpha T_{max}} \qquad (10) \quad .$$

Therefore, for example, the magnitude of the relative error at DC can be reduced to approximately .1% as follows:

$$
\begin{aligned}
& e^{-\alpha T_{max}} = .001 && \text{iff} \\
\text{iff} \quad & -\alpha T_{max} = \ell n\ .001 && \text{iff} \\
\text{iff} \quad & T_{max} = \frac{-\ell n(1000^{-1})}{\alpha} = \\
& \quad = \frac{\ell n\ 1000}{\alpha}
\end{aligned}
$$

To make the example even more specific, suppose $\alpha = 1.5 \times 10^6$. Then to reduce the DC relative error to approximately .1% equation (10) tells us that it is necessary that the truncation time t in equation (3) be no earlier than

$$T_{max} = \frac{\ell n\ 1000}{1.5 \times 10^6} = 4.6\ \mu sec$$

If the source function is truly a double exponential, then the error term $E_\omega(T_{max})$ can be calculated from equation (9) for various values of α, β, and ω. On the other hand, if the source function is only approximately the assumed double exponential then such a calculation of $E_\omega(T_{max})$ could be regarded as only a first approximation to be used in some kind of predictor-corrector process.

3. Cosine tapering.

Cosine tapering is sometimes used to drive f(t) to zero at T_{max}. In this case the analog to equation (3) used to approximate $F(\omega)$ is

$$H_\omega(T_{max}) \overset{\Delta}{=} G_\omega(rT_{max}) + H_\omega(r, T_{max}) \tag{11}$$

where r may be any real number satisfying $r \in (0,1)$, $G_\omega(rT_{max})$ is defined in equation (3), and

$$H_\omega(r,T_{max}) \triangleq \int_{rT_{max}}^{T_{max}} f(t)\cos\left[\frac{\pi(t-rT_{max})}{2T_{max}(1-r)}\right] e^{-i\omega t}\,dt \qquad (12)$$

Often r is given either the value .95 or .9 .

Consider the application of cosine tapering to the double exponential given in equation (6). For that function the $G_\omega(rT_{max})$ term in equation (11) can be had directly from equations (5) and (9) as

$$G_\omega(rT_{max}) = F(\omega) - \frac{e^{-(\alpha+i\omega)rT_{max}}}{\alpha+i\omega} + \frac{e^{-(\beta+i\omega)rT_{max}}}{\beta+i\omega} \qquad (13)$$

where $F(\omega)$ is given explicitly in equation (7). (Equation (13) is, of course, just another way of writing equation (8).) For the equation (6) double exponential, equation (12) would yield

$$H_\omega(r,T_{max}) = \int_{rT_{max}}^{T_{max}} (e^{-\alpha t}-e^{-\beta t})\cos\left[\frac{\pi(t-rT_{max})}{2T_{max}(1-r)}\right] e^{-i\omega t}\,dt =$$

$$= U_\omega(\alpha) - U_\omega(\beta) \qquad (14)$$

where

$$U_\omega(\gamma) \triangleq \int_{rT_{max}}^{T_{max}} e^{-\gamma t}\cos\left[\frac{\pi(t-rT_{max})}{2T_{max}(1-r)}\right] e^{-i\omega t}\,dt \qquad (15)$$

Appendix A begins with equation (15) and shows that

$$U_\omega(\gamma) = \frac{z[e^{-T_{max}(\gamma+i\omega)} + z(\gamma+i\omega)e^{-(\gamma+i\omega)rT_{max}}]}{z^2(\gamma+i\omega)^2+1} \qquad (16)$$

where

$$z \triangleq \frac{2T_{max}(1-r)}{\pi} .$$

Combining equations (11), (13), and (14), we then have

$$H_\omega(T_{max}) = \frac{e^{-(\beta+i\omega)rT_{max}}}{\beta+i\omega} - \frac{e^{-(\alpha+i\omega)rT_{max}}}{\alpha+i\omega} + $$

$$+ F(\omega) + U_\omega(\alpha) - U_\omega(\beta) \qquad (17) .$$

4. Linear tapering.

Cosine tapering is not the only kind of tapering used. Sometimes a linear function of t is subtracted from f(t) to make it zero at T_{max}. Such linear tapering is especially appropriate when the signal is coming from a device like an integrator since integrators are prone to drift approximately linearly in time.

In the case of linear tapering f(t) is approximated by

$$v(t) = f(t) - \frac{f(T_{max})}{T_{max}} t .$$

F(ω) is then approximated by

$$V_\omega(T_{max}) = \int_0^{T_{max}} v(t)\, e^{-i\omega t}\, dt =$$

$$= \int_0^{T_{max}} [f(t) - bt]\, e^{-i\omega t}\, dt =$$

$$= \int_0^{T_{max}} f(t)\, e^{-i\omega t}\, dt - b \int_0^{T_{max}} t\, e^{-i\omega t}\, dt \qquad (18)$$

where

$$b \triangleq \frac{f(T_{max})}{T_{max}} .$$

Integrating by parts (or using tables) shows

$$\int t\, e^{-i\omega t}\, dt = \frac{e^{-i\omega t}}{\omega^2}(1+i\omega t) .$$

Applying this result and equation (3) to equation (18) yields

$$V_\omega(T_{max}) = G_\omega(T_{max}) - \frac{be^{-i\omega t}}{\omega^2}(1+i\omega t)\Big|_{t=0}^{T_{max}} =$$

$$= G_\omega(T_{max}) - \frac{b}{\omega^2}[e^{-i\omega T_{max}}(1+i\omega T_{max}) - 1] =$$

$$= G_\omega(T_{max}) + \frac{f(T_{max})}{\omega^2 T_{max}}[1-e^{-i\omega T_{max}}(1+i\omega T_{max})] \qquad (19) .$$

Analogous to equation (4) we in this case define the error

$$D_\omega(T_{max}) \triangleq F(\omega) - V_\omega(T_{max}) .$$

Applying equation (5), this becomes

$$D_\omega(T_{max}) = G_\omega(T_{max}) + E_\omega(T_{max}) - V_\omega(T_{max}) .$$

Combining this with equation (19) then yields

$$D_\omega(T_{max}) = E_\omega(T_{max}) - \frac{f(T_{max})}{\omega^2 T_{max}}[1-e^{-i\omega T_{max}}(1+i\omega T_{max})] =$$

$$= E_\omega(T_{max}) + \frac{f(T_{max})}{\omega^2 T_{max}}[e^{-i\omega T_{max}}(1+i\omega T_{max}) - 1] \qquad (20)$$

For the double exponential of equation (6), equation (9) makes equation (20) become

$$D_\omega(T_{max}) = \frac{e^{-(\alpha+i\omega)T_{max}}}{\alpha+i\omega} - \frac{e^{-(\beta+i\omega)T_{max}}}{\beta+i\omega} +$$

$$+ \frac{f(T_{max})}{\omega^2 T_{max}}[e^{-i\omega T_{max}}(1+i\omega T_{max}) -1]$$

5. Summary.

We now have an equation for the frequency domain errors induced by time truncation of a double exponential signal (equation (9)). We also have the equations for errors due to time truncation when data has been "tapered" by cosine (equation (17)) or linearly (equation (20)) to reduce the frequency domain errors.

II. Numerical Results

1. Presentation.

To generate numerical results, an arbitrary double exponential was picked ($f(t) = e^{-1.5x10^6 t} - e^{-2.6x10^8 t}$) and the magnitude of the error term as a percentage of the known value of the analytic Fourier transform of the double exponential was plotted. The plots are contour plots, in truncation time-frequency data space, e.g., the vertical axis is truncation time in seconds and the horizontal axis is frequency in Hertz. The plots are arranged in sets for which different contours were selected. The scale is a pseudo-logarithmic scale in that it is linear within each decade.

These plots are intended to show the relative effects of the various kinds of tapering. As such, they do suffer from some computational inadequacies (e.g., only small efforts were made to avoid "small difference of large numbers" problems). This does not mean that the plots are numerically useless, but extracting absolute values from these plots should be limited to ± 1/4 inch and values of error greater than 1%.

Each set of plots is arranged in the following order:

1. Pure truncation
2. Cosine taper of the last 5% of the time data
3. Cosine taper of the last 10% of the time data
4. Cosine taper of the last 15% of the time data
5. Subtraction of a ramp function to force the time data to zero at the last point.

The plots in Appendix B are composite plots which present all the contours for a given taper. As one would expect, cosine tapering and subtracting a linear function both help at the high frequency end. It is felt that the dramatic decrease in error magnitude upon subtracting a linear term is due to the relatively small discontinuity in the derivative of the time function. This leads one to the speculation that perhaps it would help to invent a tapering function that would force the tail of the function to be zero and the derivative of the function to be equal to the derivative at the start of the wave (i.e., $\beta-\alpha$ for the double exponential).

The particular cosine tapering used for these plots spanned a range of cos 0 to cos $\frac{\pi}{2}$ (see equation 12). This can result in a sharp discontinuity of the time derivative at the window time T_{max}. However, cosine tapering from cos 0 to cos π is sometimes used. This would reduce the discontinuity in the derivative at the window time and may very well be the equal of subtracting a linear term in its enhancement of accuracy.

The plots in Appendix C were made by letting the contour plotting routine pick four equally spaced contours in the window time-frequency plane. The contours do not have the same values since there are different maximum values for the error depending on the taper used. It is interesting to note that subtraction of a linear term results in much less maximum error than the other tapering techniques.

The contours in Appendix D are of the error values 80% through 130% in steps of 10%, and the contours in Appendix E are of the error values 20% through 80% in steps of 20%.

Even 5% cosine tapering improves the high frequency data remarkably. Again, subtracting a linear term appears superior. It does introduce slightly higher errors in the low frequency range but results in much higher accuracy in the high frequency range.

The contours for Appendix F are of the error values 2% through 8% in steps of 2%. These are the most interesting of the plots because they represent values of error which can be tolerated.

A 5% cosine taper helps the accuracy of the data quite a bit; a 10% taper helps the data a fair amount more; and a 15% taper is just a little bit better than the 10% taper, indicating that the cosine taper is approaching a limit. The subtraction of a linear term results in far better data at the high end of the spectrum and only slightly degraded data at the low end.

The plots of Appendix G were run as a matter of curiosity and are of the .1% error contour. This data cannot be trusted too much in terms of absolute values (e.g., for pure truncation the contour should be at 4.6 μsecs instead of 10 μsecs). However, the comparison between plots is still valid and indicates the superiority of the use of linear tapering over this type of cosine tapering for this waveform.

2. Conclusions.

The calculations described here were intended to give some insight into the errors induced by time truncation and how effective two tapering techniques are. For the specific double exponential $f(t) = e^{-1.5\times10^6 t} - e^{-2.6\times10^8 t}$, subtracting a linear term ($f = bt$) (thereby forcing the value to zero at the truncation time) is superior to the particular cosine tapering technique described. However, extending that conclusion (to indicate that subtracting a linear term is superior for arbitrary waveshape or is superior to other tapering procedures) is not warranted based on the calculations presented here.

Appendix A: Derivation of $U_\omega(\gamma)$.

From the definition of $U_\omega(\gamma)$ in equation (15), we have

$$U_\omega(\gamma) = \int_{rT_{max}}^{T_{max}} e^{-(\gamma+i\omega)t} \cos\left[\frac{\pi(t-rT_{max})}{2T_{max}(1-r)} \right] dt \qquad (A1)$$

Let a (for "argument") be defined by

$$a \triangleq \frac{\pi(t-rT_{max})}{2T_{max}(1-r)} \qquad (A2)$$

Solving this equation for t in terms of a yields

$$t = \frac{2aT_{max}(1-r)}{\pi} + rT_{max} =$$

$$= za + rT_{max} \qquad (A3)$$

where

$$z \triangleq \frac{2T_{max}(1-r)}{\pi} \qquad (A4)$$

Equation (A3) implies

$$dt = zda$$

By equation (A2),

$$t = rT_{max} \quad \text{iff} \quad a = 0 \quad \text{and}$$

$$t = T_{max} \quad \text{iff} \quad a = \frac{\pi}{2} \quad .$$

Applying equation (A2) and the two preceding sentences to equation (A1) yields

$$U_\omega(\gamma) = \int_0^{\frac{\pi}{2}} e^{-(\gamma+i\omega)(za+rT_{max})} \cos(a)\, z\, da =$$

$$= z \int_0^{\frac{\pi}{2}} e^{-(\gamma+i\omega)za}\, e^{-(\gamma+i\omega)rT_{max}} \cos(a)\, da =$$

$$= ze^{-(\gamma+i\omega)rT_{max}} \int_0^{\frac{\pi}{2}} e^{-(\gamma+i\omega)za} \cos(a)\, da =$$

$$= ze^{-(\gamma+i\omega)rT_{max}} \int_0^{\frac{\pi}{2}} e^{ba} \cos(a)\, da$$

where

$$b \triangleq -(\gamma+i\omega)z \qquad \text{(A5)} \quad .$$

Integrating by parts twice (or using tables) yields

$$U_\omega(\gamma) = ze^{-(\gamma+i\omega)rT_{max}} \left[\frac{e^{ba}}{b^2+1} (b \cos a + \sin a) \right]_0^{\frac{\pi}{2}} =$$

$$= ze^{-(\gamma+i\omega)rT_{max}} \left[\frac{e^{\frac{\pi b}{2}}}{b^2+1} (b \cdot 0 + 1) - \frac{e^0}{b^2+1} (b \cdot 1 + 0) \right] =$$

$$= \frac{z(e^{\frac{\pi b}{2}} - b)}{(b^2+1)e^{(\gamma+i\omega)rT_{max}}} \quad .$$

Using equation (A5) to eliminate b then yields

$$U_\omega(\gamma) = \frac{z[e^{-\frac{\pi}{2}z(\gamma+i\omega)} + z(\gamma+i\omega)]}{[z^2(\gamma+i\omega)^2+1]e^{(\gamma+i\omega)rT_{max}}}$$

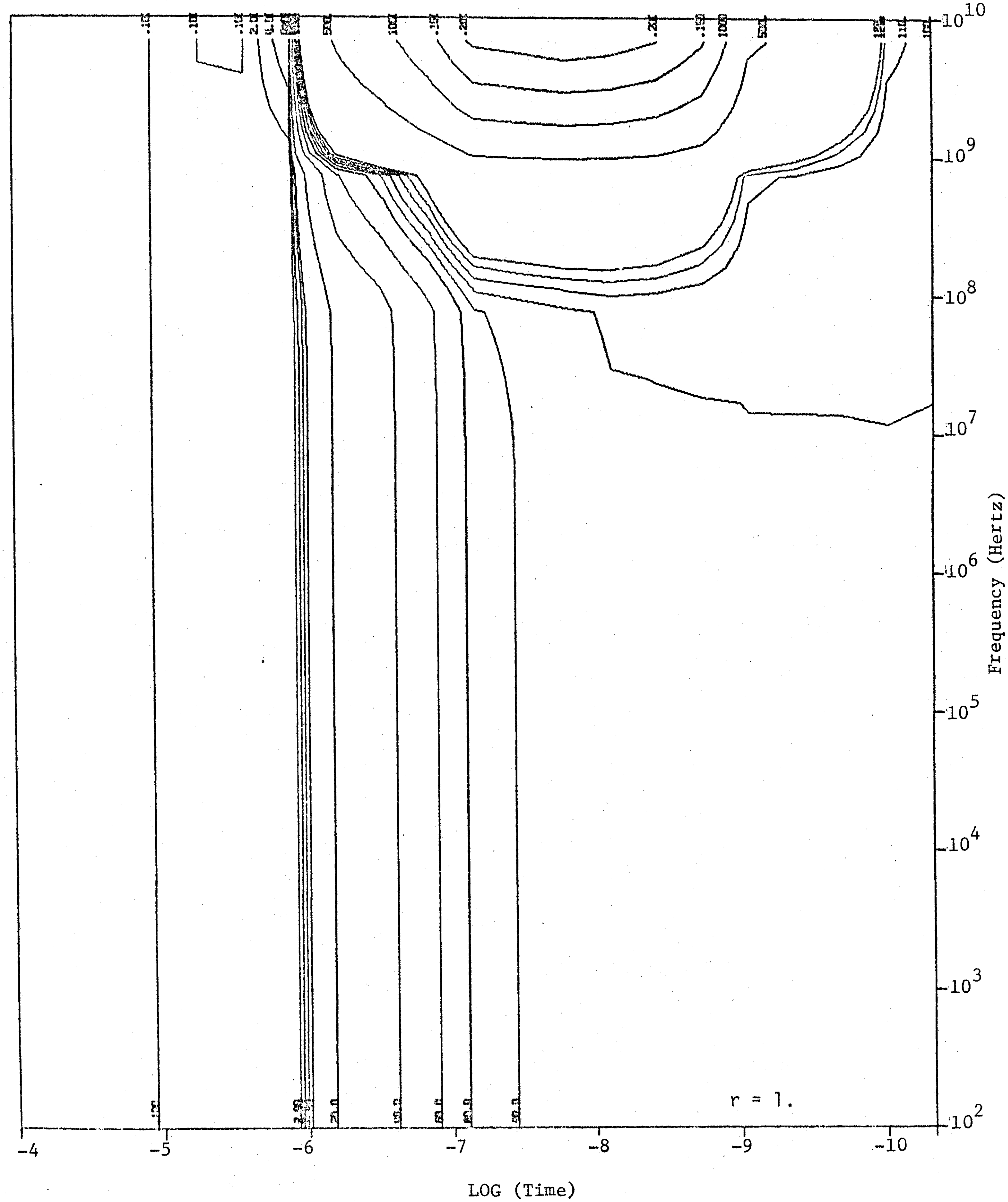
Applying equation (A4) to the numerator exponent makes this become

$$U_\omega(\gamma) = \frac{z[e^{-T_{max}(1-r)(\gamma+i\omega)} + z(\gamma+i\omega)]}{[z^2(\gamma+i\omega)^2+1]e^{(\gamma+i\omega)rT_{max}}} =$$

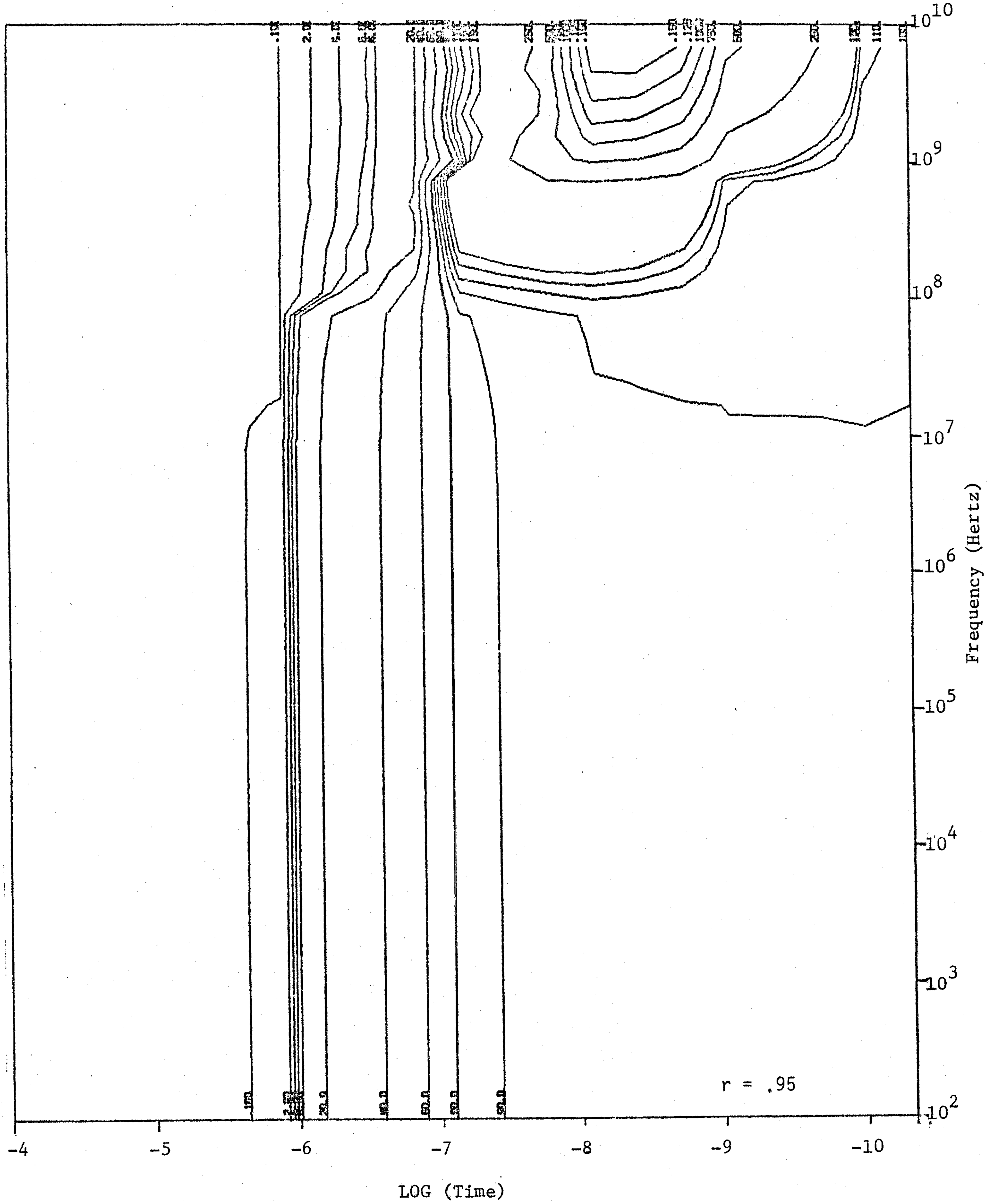
$$= \frac{z[e^{-T_{max}(\gamma+i\omega)} + z(\gamma+i\omega)e^{-(\gamma+i\omega)rT_{max}}]}{z^2(\gamma+i\omega)^2 + 1}$$

APPENDIX B

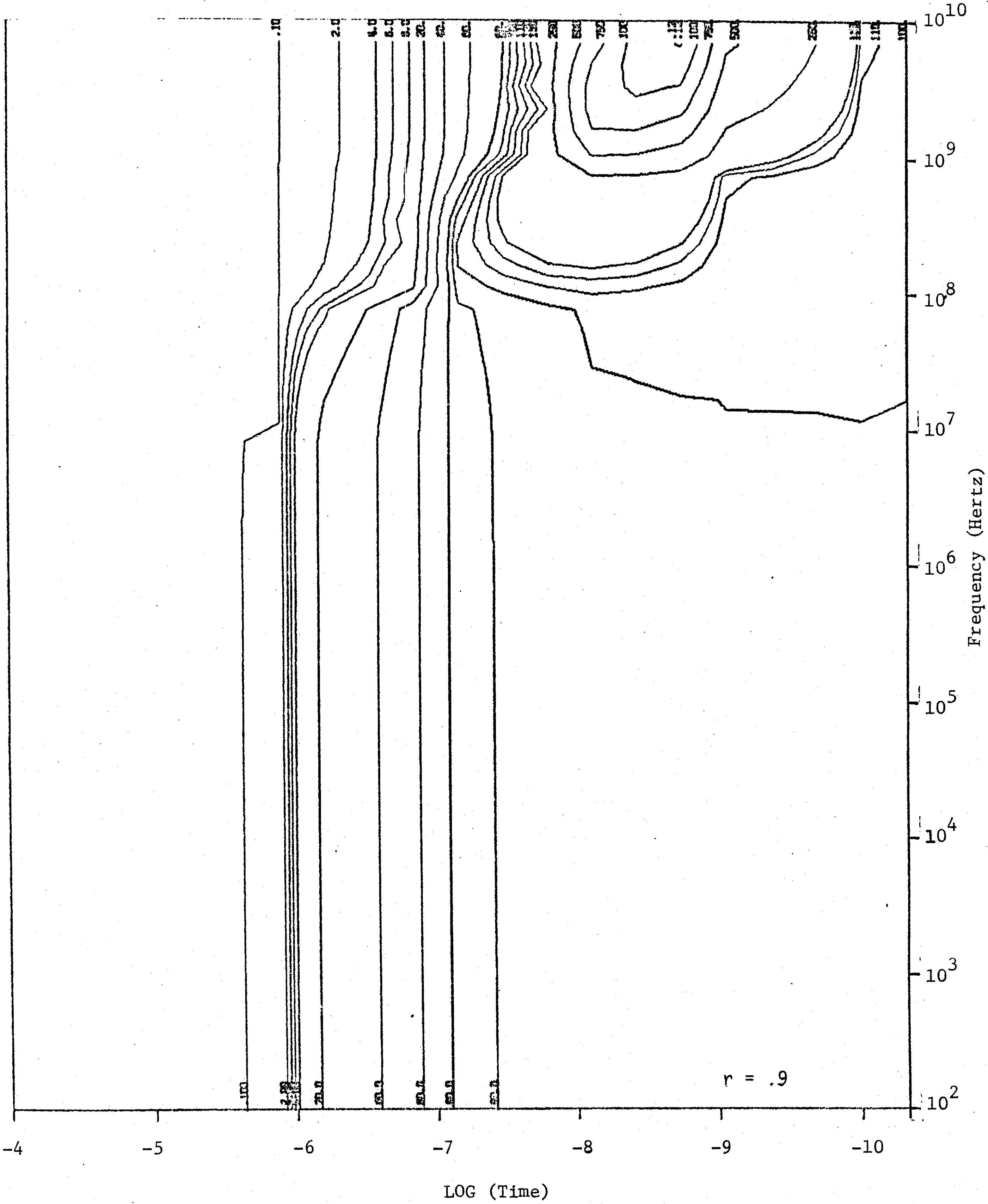
Composite Plots of Percentage Magnitude Error
(with Cosine Taper or Linear Taper).



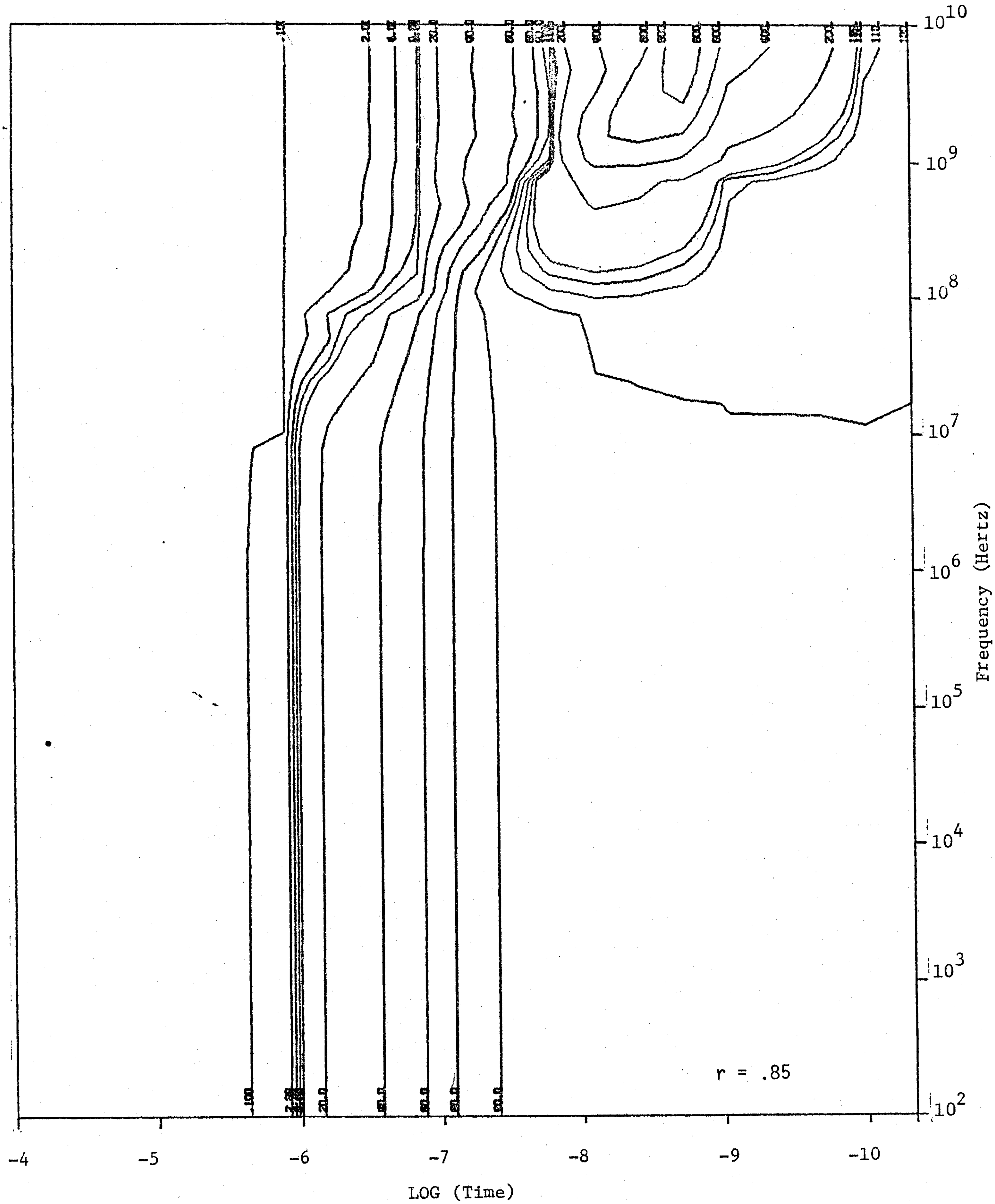
10^{10}
10^9
10^8
10^7
10^6
10^5
10^4
10^3
10^2
Frequency (Hertz)
r = 1.
-4
-5
-6
-7
-8
-9
-10
LOG (Time)



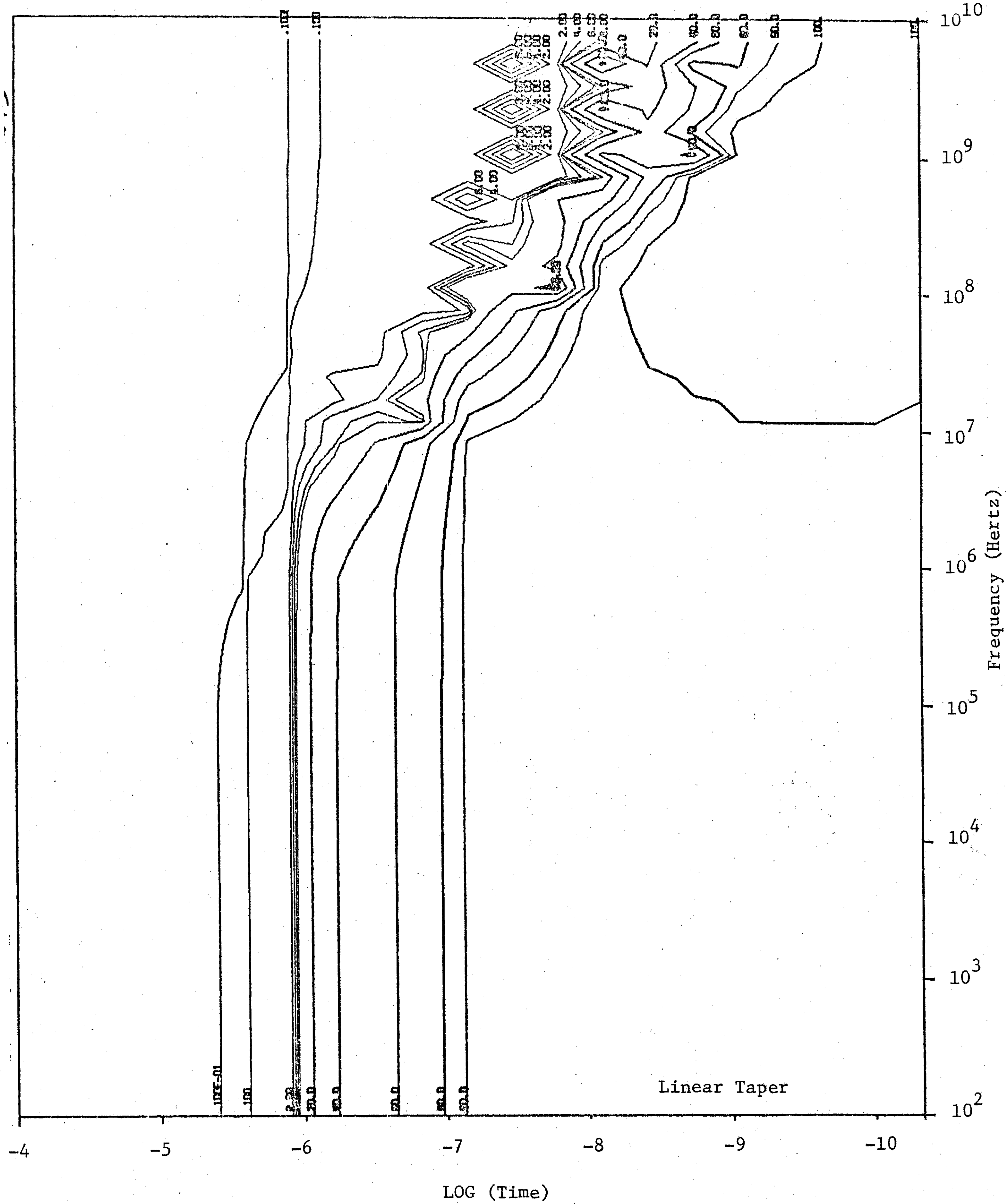
10^{10}
10^{9}
10^{8}
10^{7}
10^{6}
10^{5}
10^{4}
10^{3}
10^{2}
Frequency (Hertz)
r = .95
-4
-5
-6
-7
-8
-9
-10
LOG (Time)



10^10
10^9
10^8
10^7
10^6
10^5
10^4
10^3
10^2
Frequency (Hertz)
r = .9
-4
-5
-6
-7
-8
-9
-10
LOG (Time)

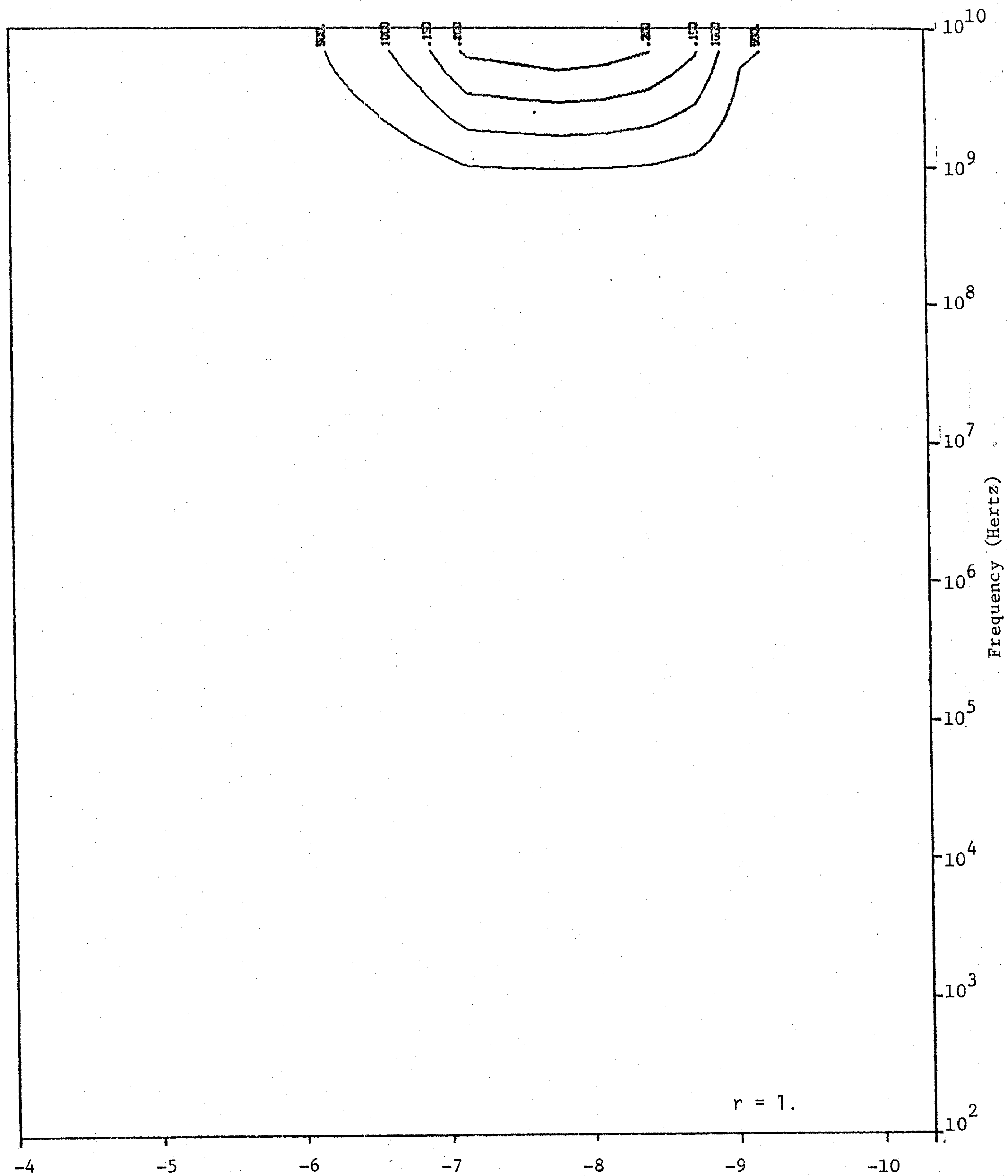


Frequency (Hertz)
10^{10}
10^{9}
10^{8}
10^{7}
10^{6}
10^{5}
10^{4}
10^{3}
10^{2}
r = .85
-4
-5
-6
-7
-8
-9
-10
LOG (Time)

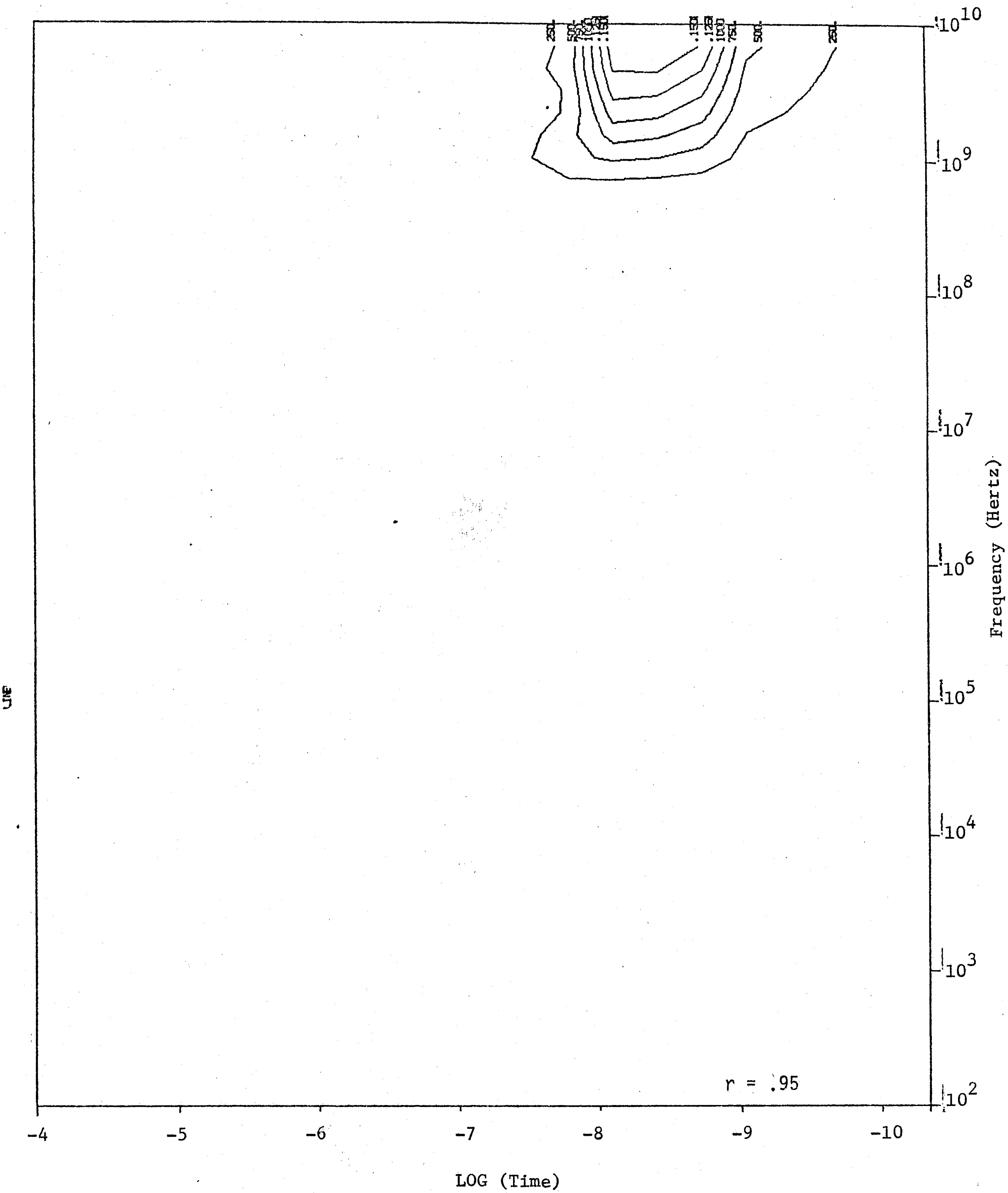

Linear Taper
LOG (Time)
-4
-5
-6
-7
-8
-9
-10
Frequency (Hertz)
10^{2}
10^{3}
10^{4}
10^{5}
10^{6}
10^{7}
10^{8}
10^{9}
10^{10}

APPENDIX C

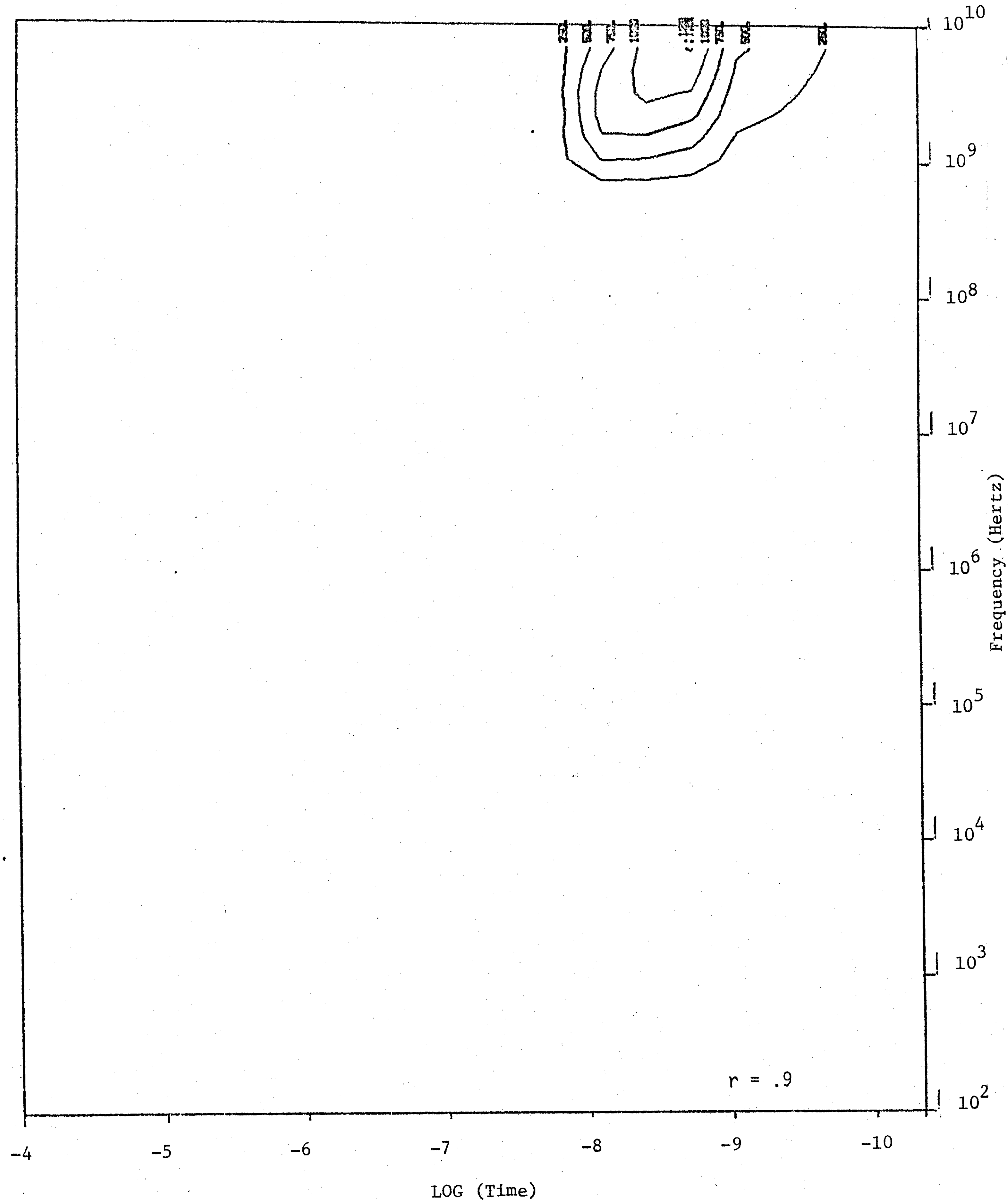
Four Equally Spaced Contours of Percentage Magnitude Errors.



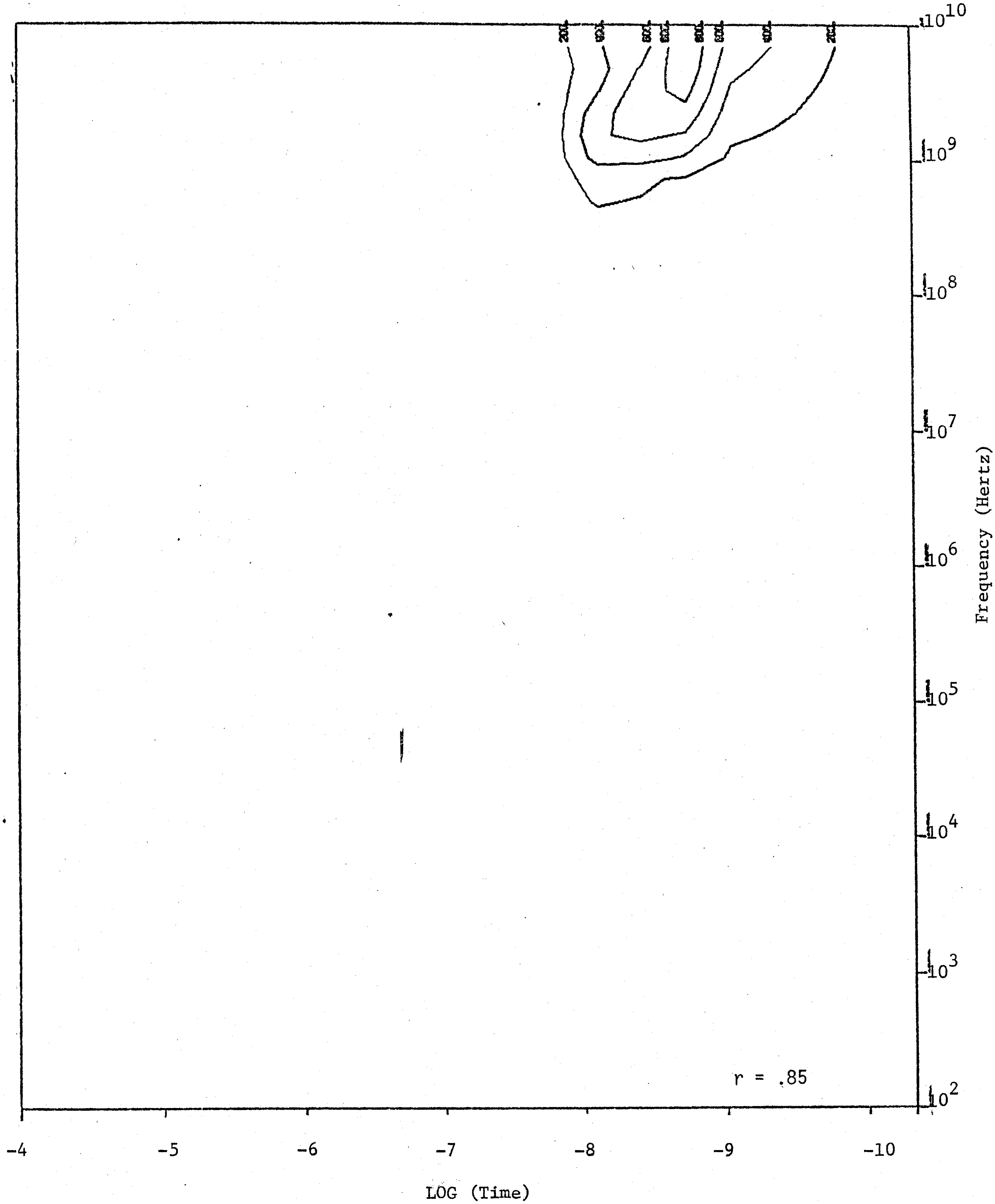
10^{10}
10^9
10^8
10^7
10^6
10^5
10^4
10^3
10^2
Frequency (Hertz)
r = 1.
-4
-5
-6
-7
-8
-9
-10



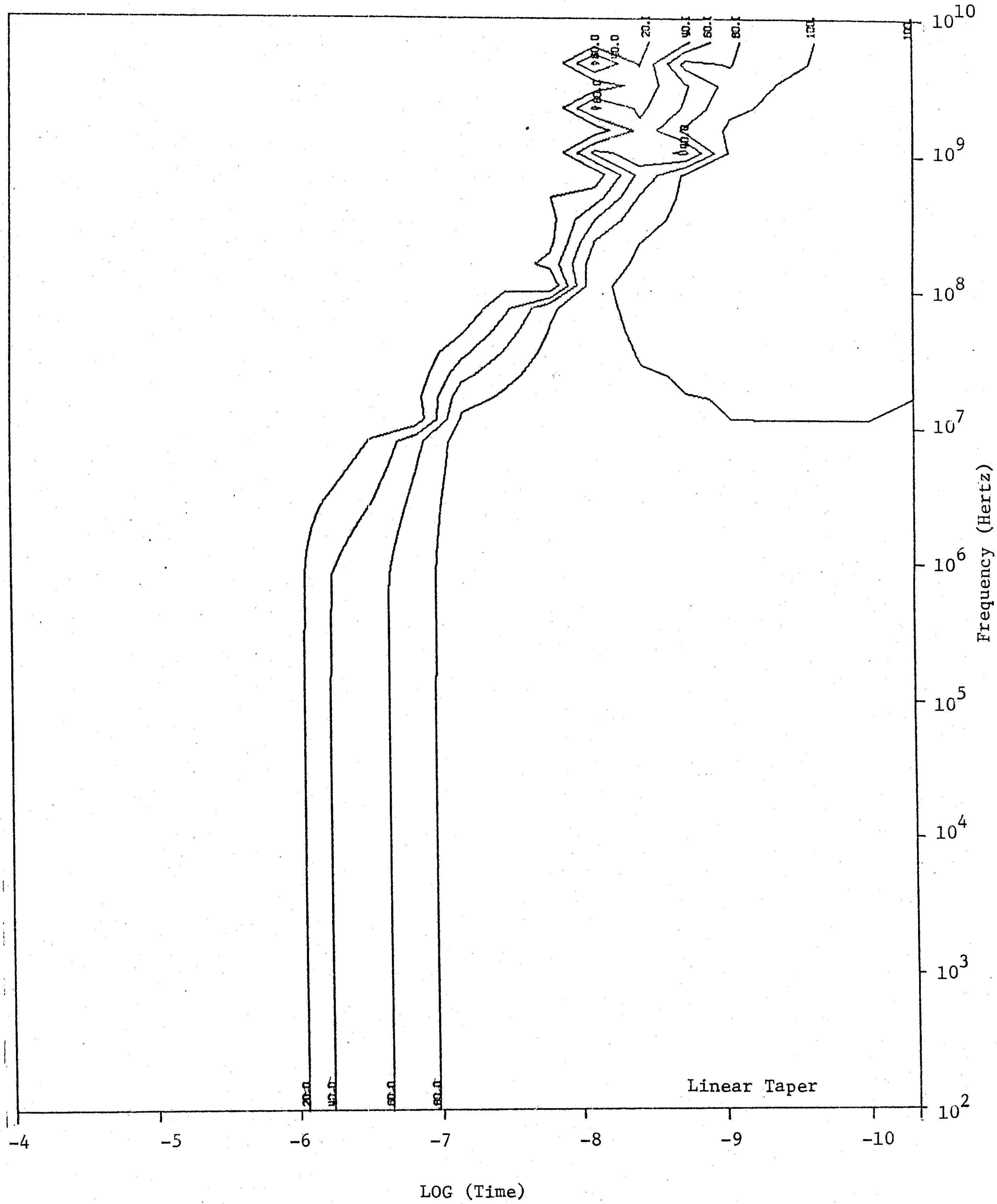
10^{10}
10^9
10^8
10^7
10^6
10^5
10^4
10^3
10^2
Frequency (Hertz)
r = .95
-4
-5
-6
-7
-8
-9
-10
LOG (Time)



r = .9
LOG (Time)
-4
-5
-6
-7
-8
-9
-10
Frequency (Hertz)
10^{10}
10^9
10^8
10^7
10^6
10^5
10^4
10^3
10^2

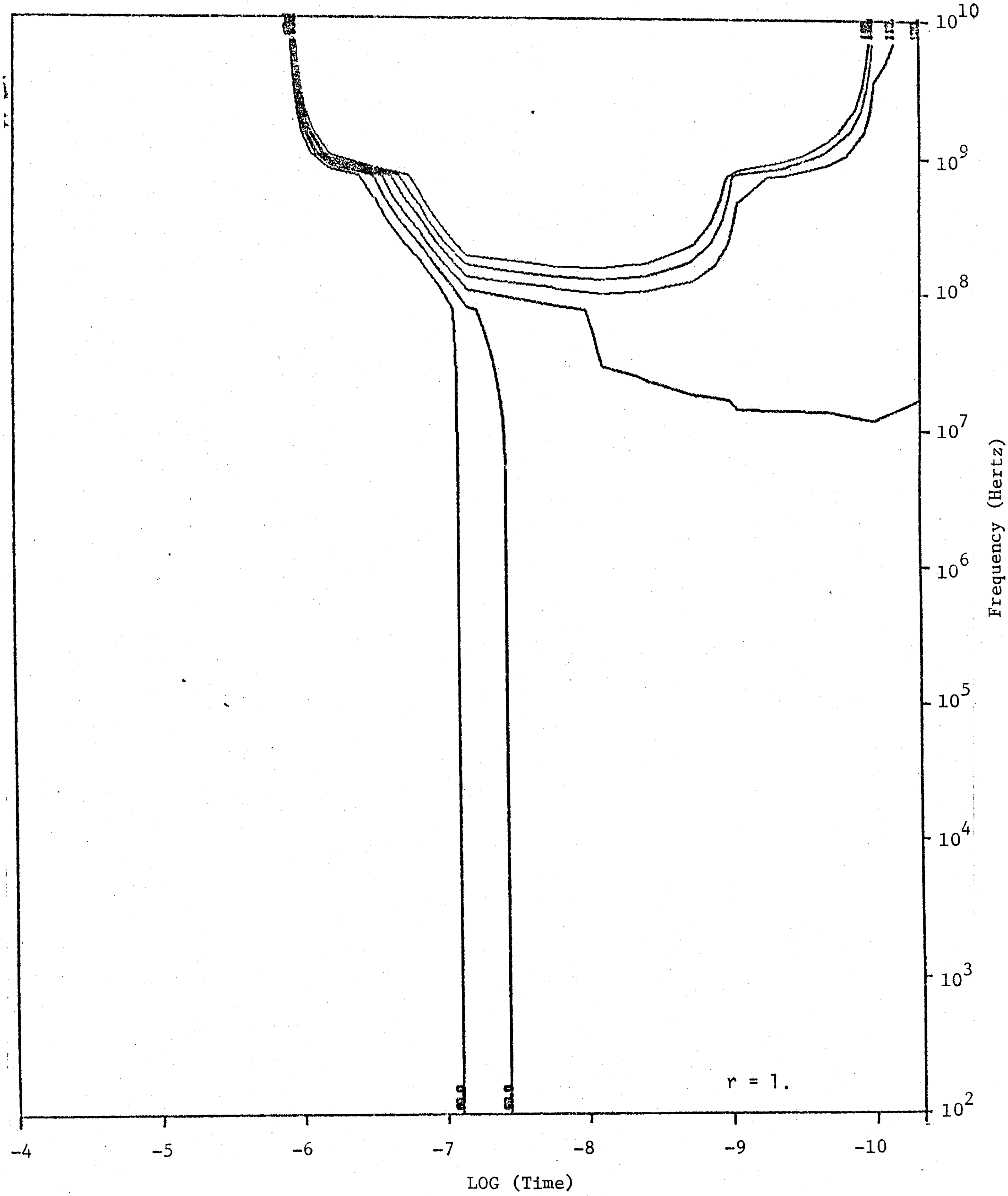


10^{10}
10^9
10^8
10^7
10^6
10^5
10^4
10^3
10^2
Frequency (Hertz)
r = .85
-4
-5
-6
-7
-8
-9
-10
LOG (Time)

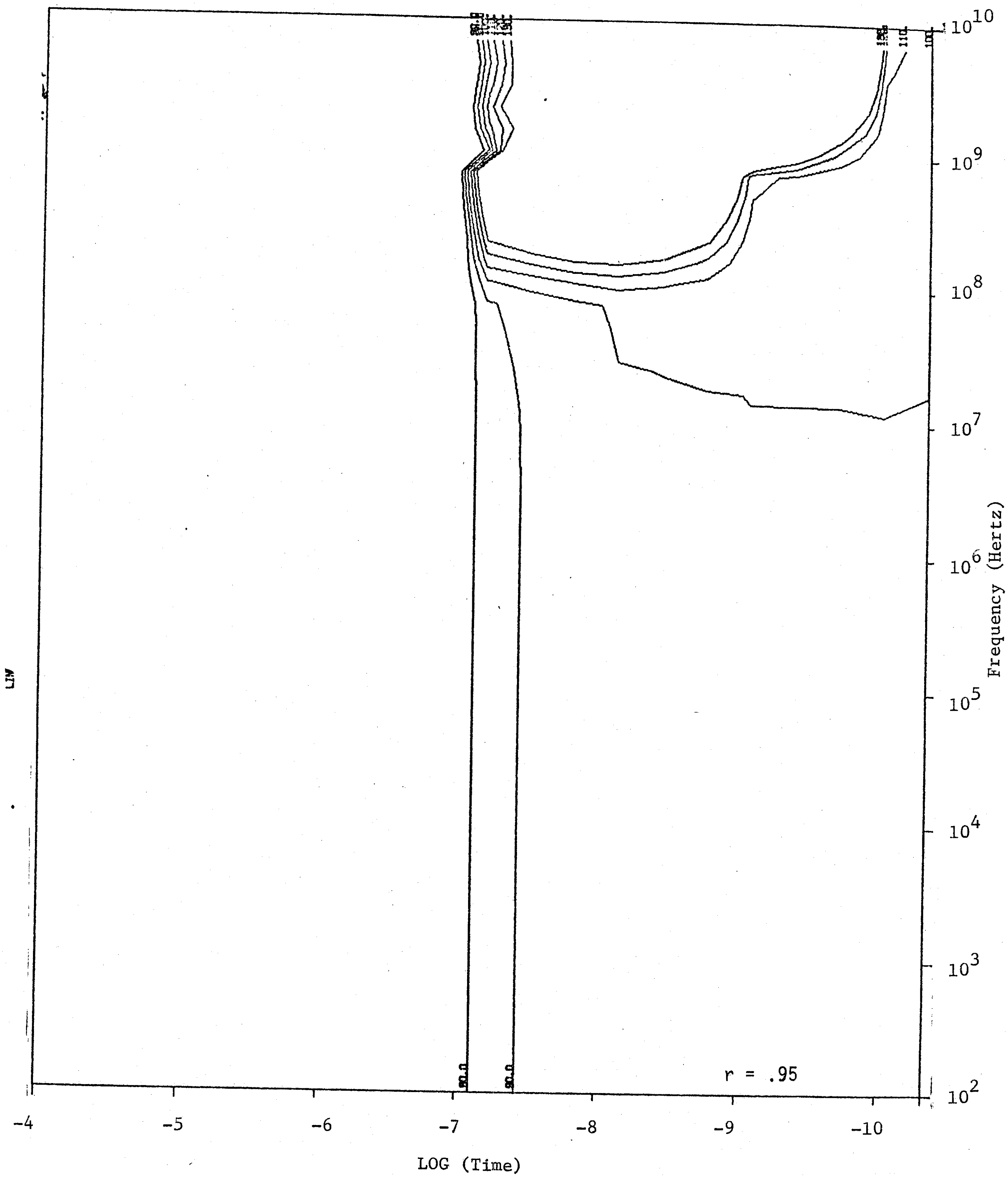

Linear Taper
LOG (Time)
Frequency (Hertz)
-4
-5
-6
-7
-8
-9
-10
10^2
10^3
10^4
10^5
10^6
10^7
10^8
10^9
10^{10}
20.0
40.0
60.0
80.0

APPENDIX D

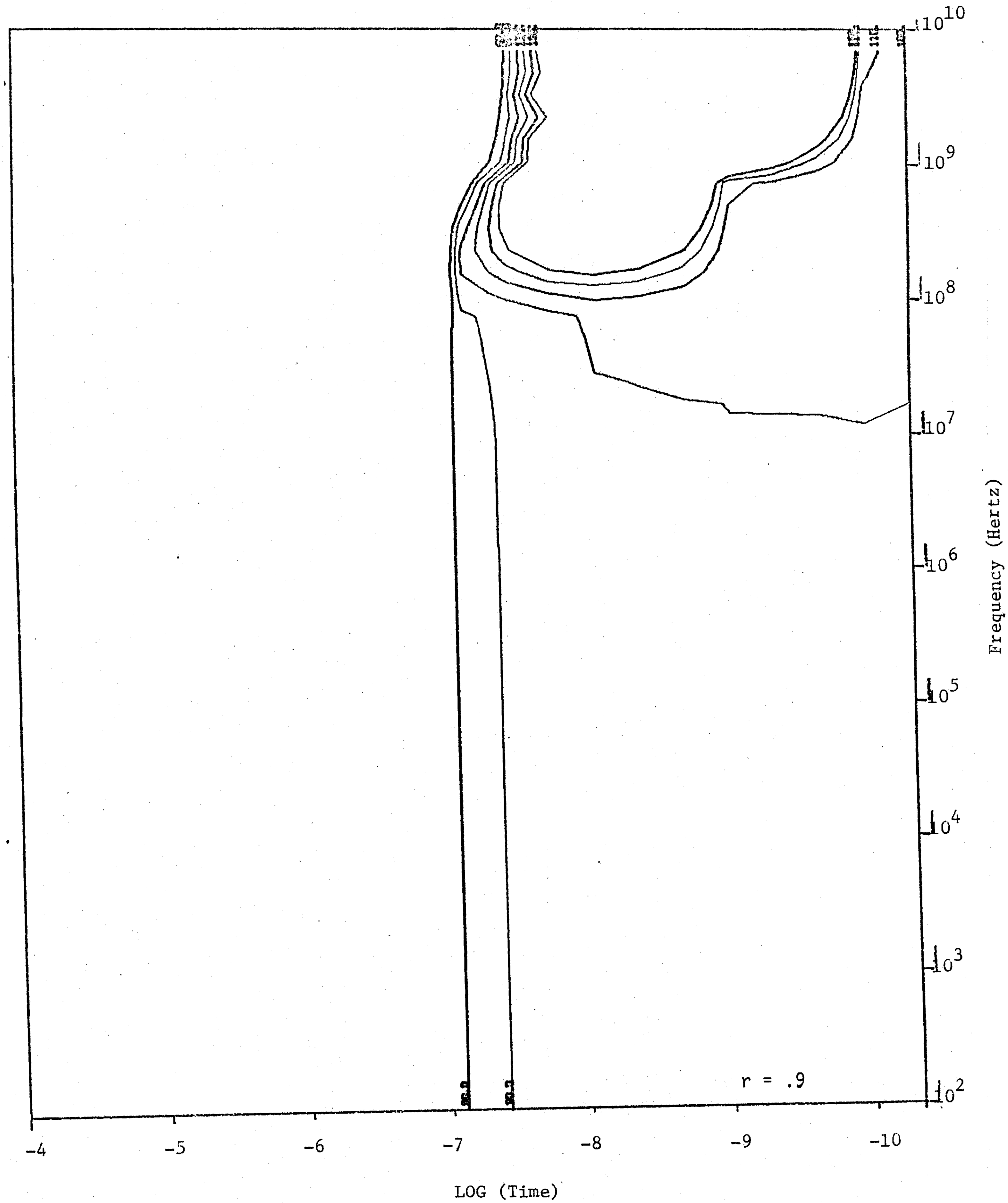
Magnitude Errors of 80% to 130% in Steps of 10%.



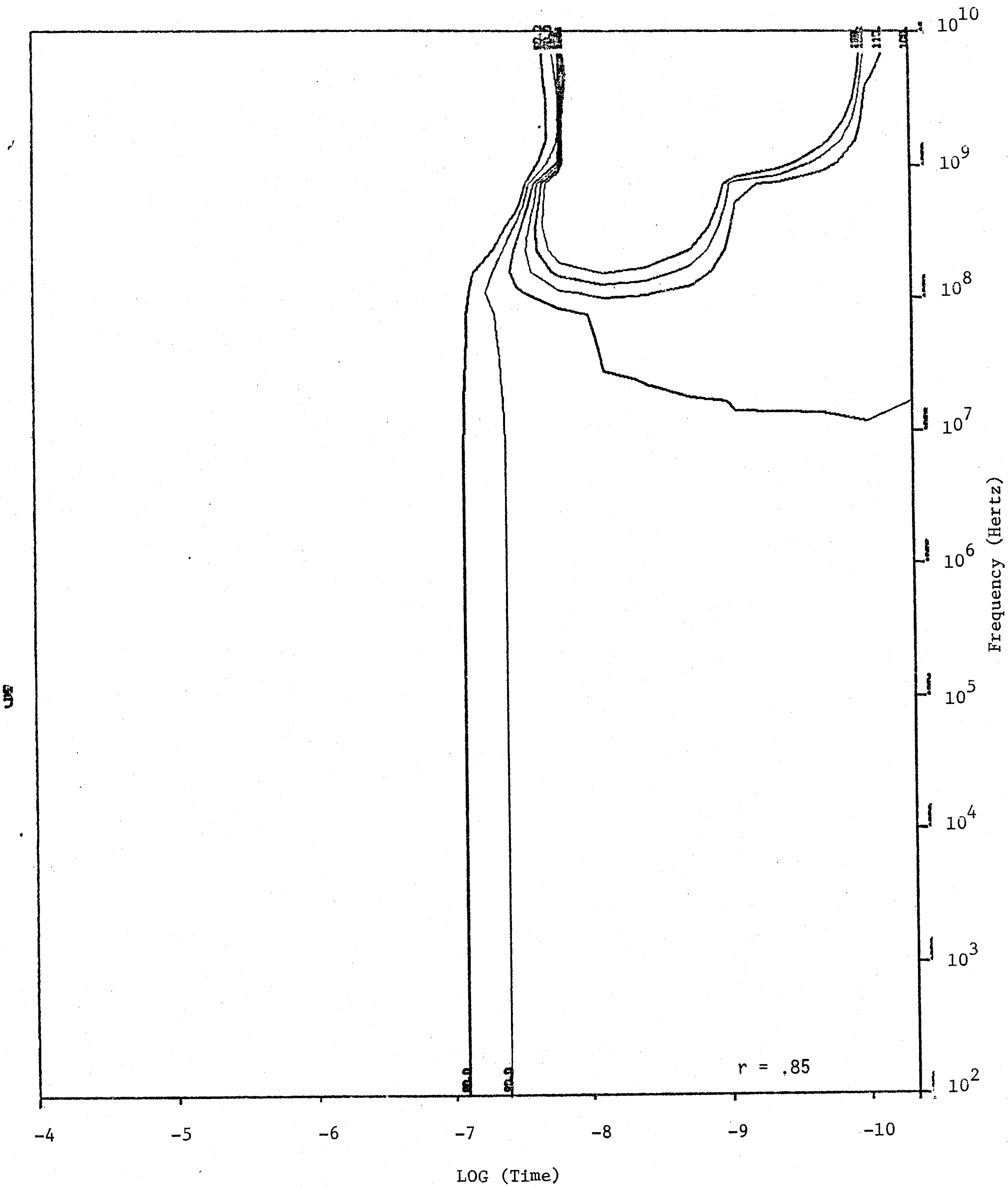
r = 1.
-4
-5
-6
-7
-8
-9
-10
LOG (Time)
10^{10}
10^9
10^8
10^7
10^6
10^5
10^4
10^3
10^2
Frequency (Hertz)



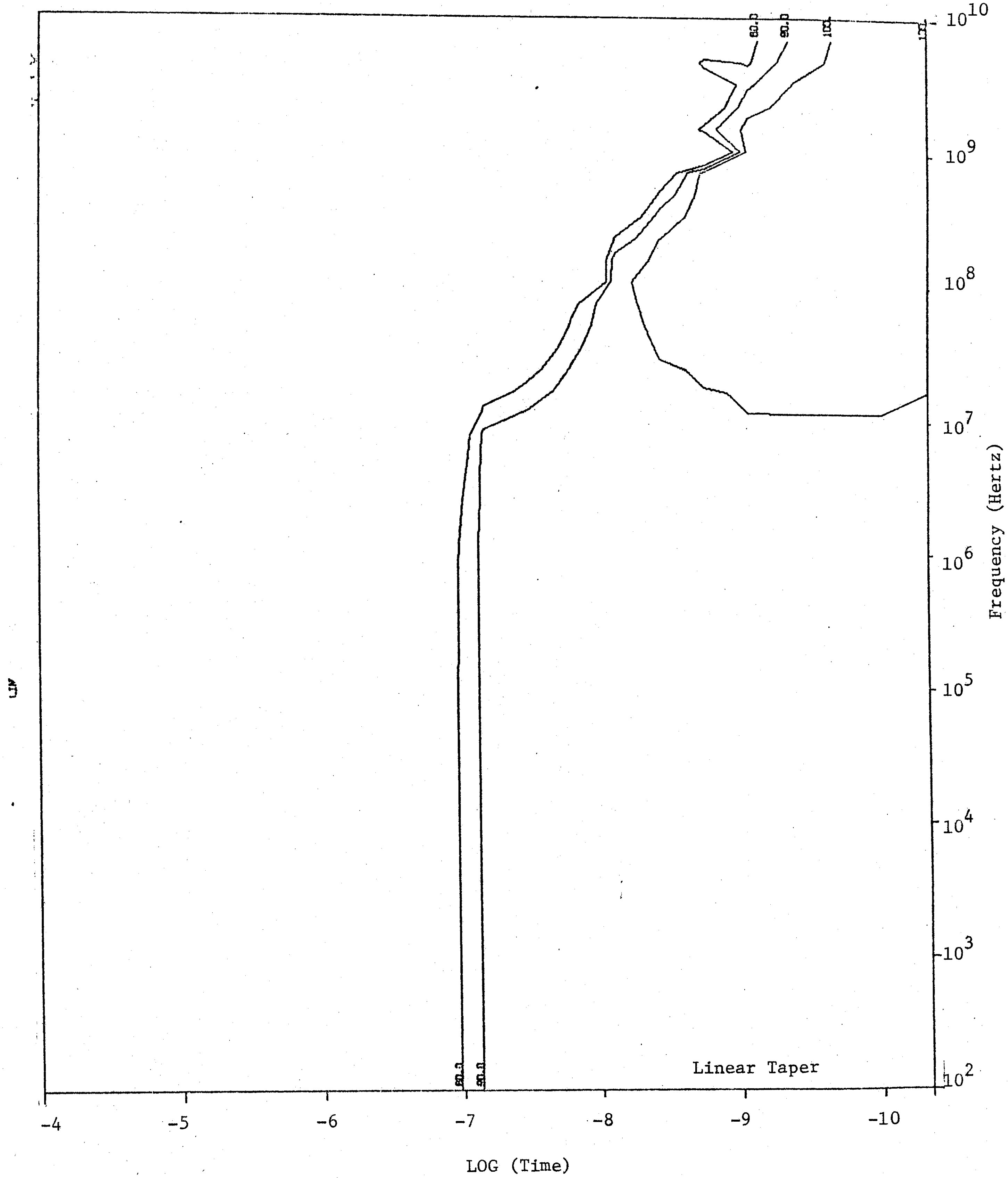
r = .95
90.0
90.0
10^{10}
10^9
10^8
10^7
10^6
10^5
10^4
10^3
10^2
Frequency (Hertz)
-4
-5
-6
-7
-8
-9
-10
LOG (Time)



10^{10}
10^9
10^8
10^7
Frequency (Hertz)
10^6
10^5
10^4
10^3
10^2
r = .9
-4
-5
-6
-7
-8
-9
-10
LOG (Time)



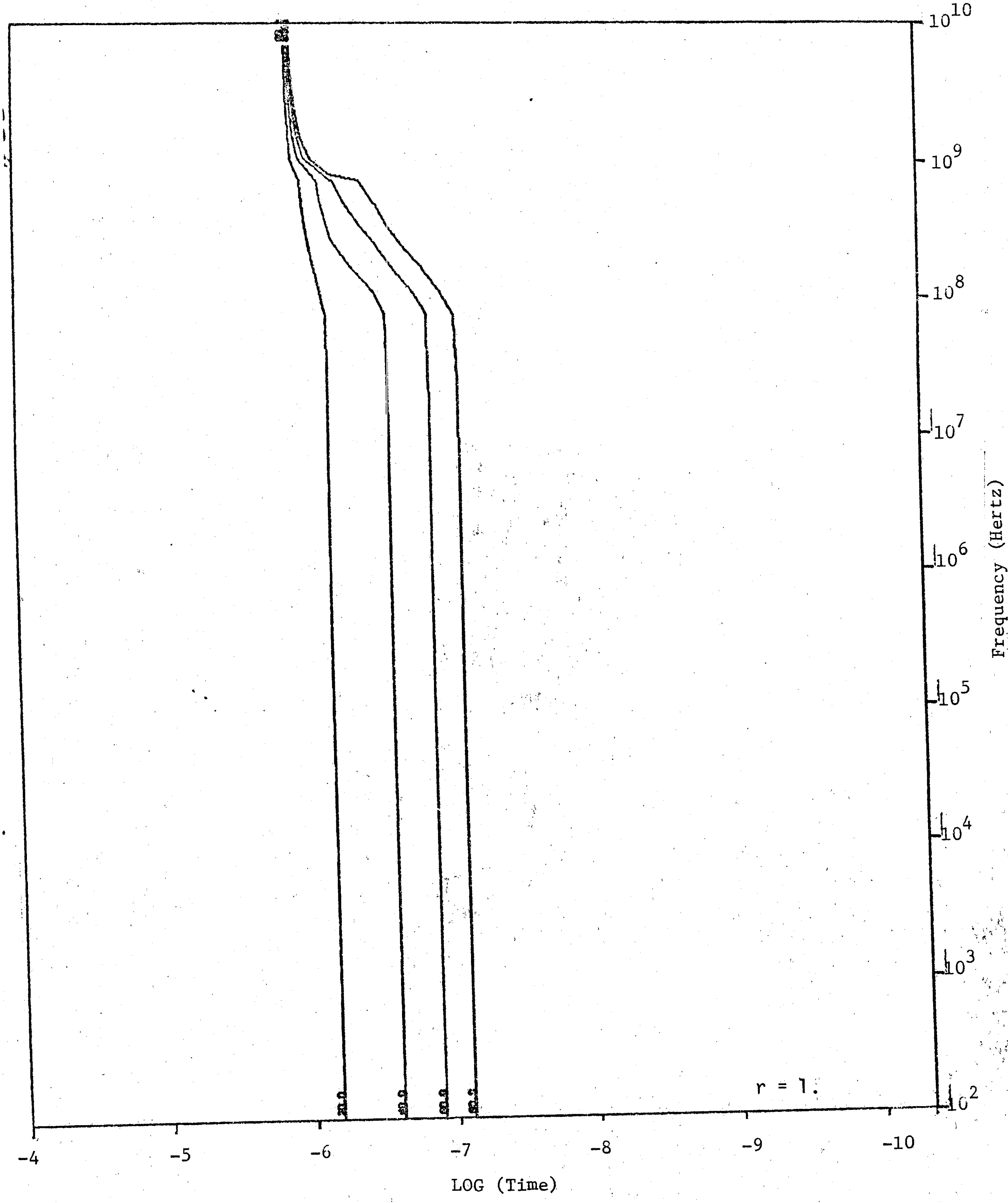
10^{10}
10^{9}
10^{8}
10^{7}
10^{6}
10^{5}
10^{4}
10^{3}
10^{2}
Frequency (Hertz)
r = .85
-4
-5
-6
-7
-8
-9
-10
LOG (Time)



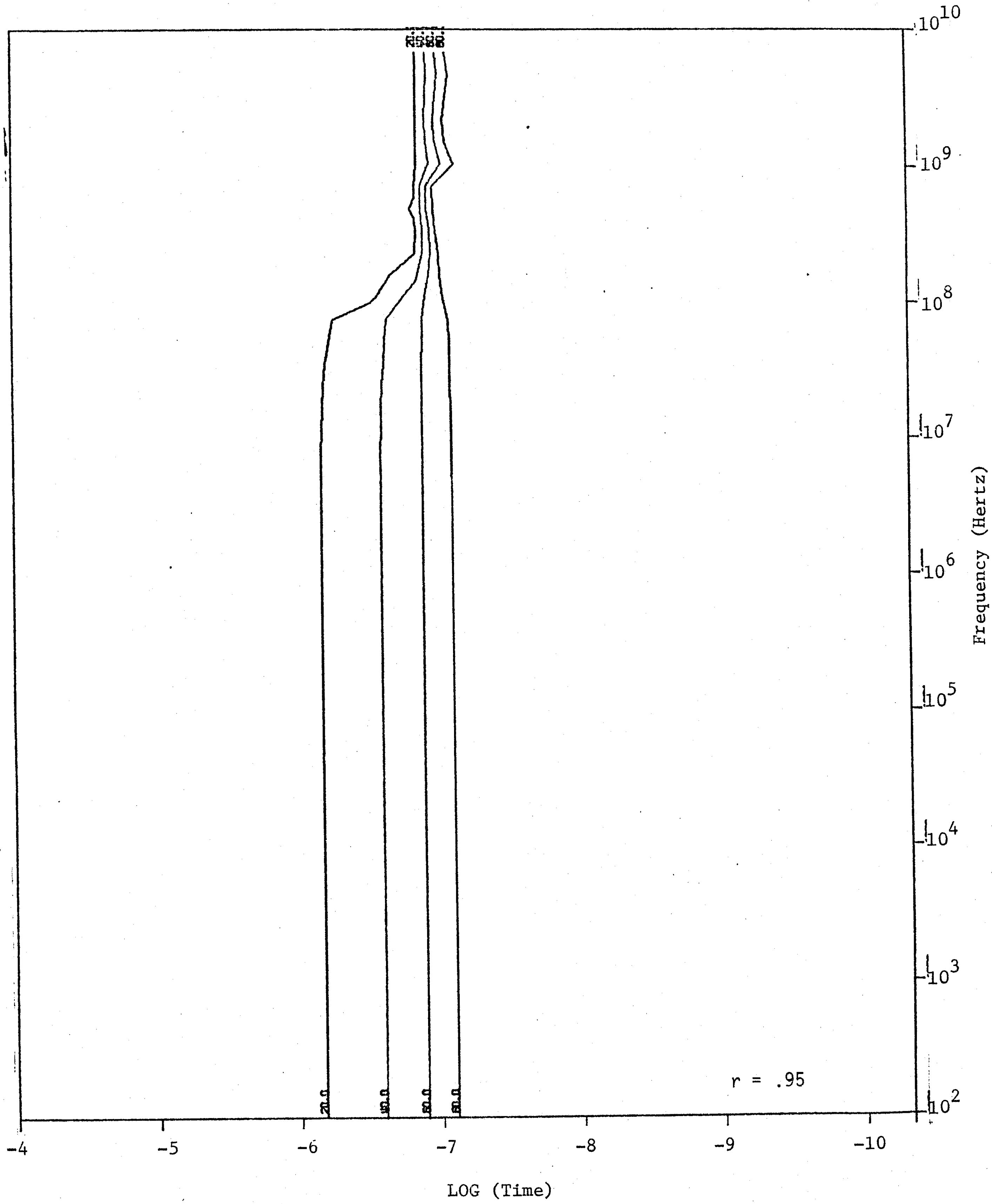
80.0
90.0
100
120
Linear Taper
-4
-5
-6
-7
-8
-9
-10
LOG (Time)
10^{10}
10^{9}
10^{8}
10^{7}
10^{6}
10^{5}
10^{4}
10^{3}
10^{2}
Frequency (Hertz)

APPENDIX E

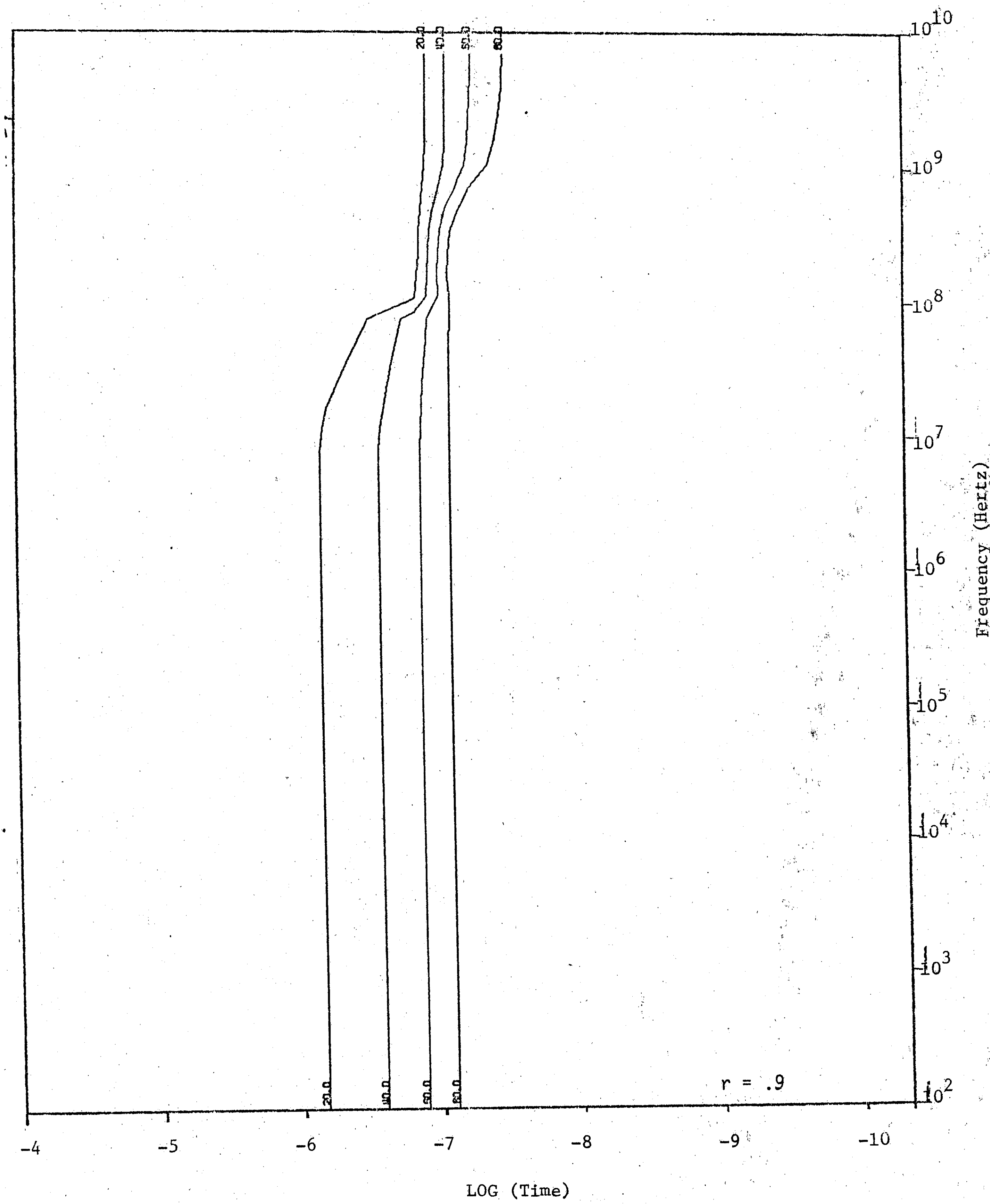
Magnitude Errors of 20% to 80% in Steps of 20%.



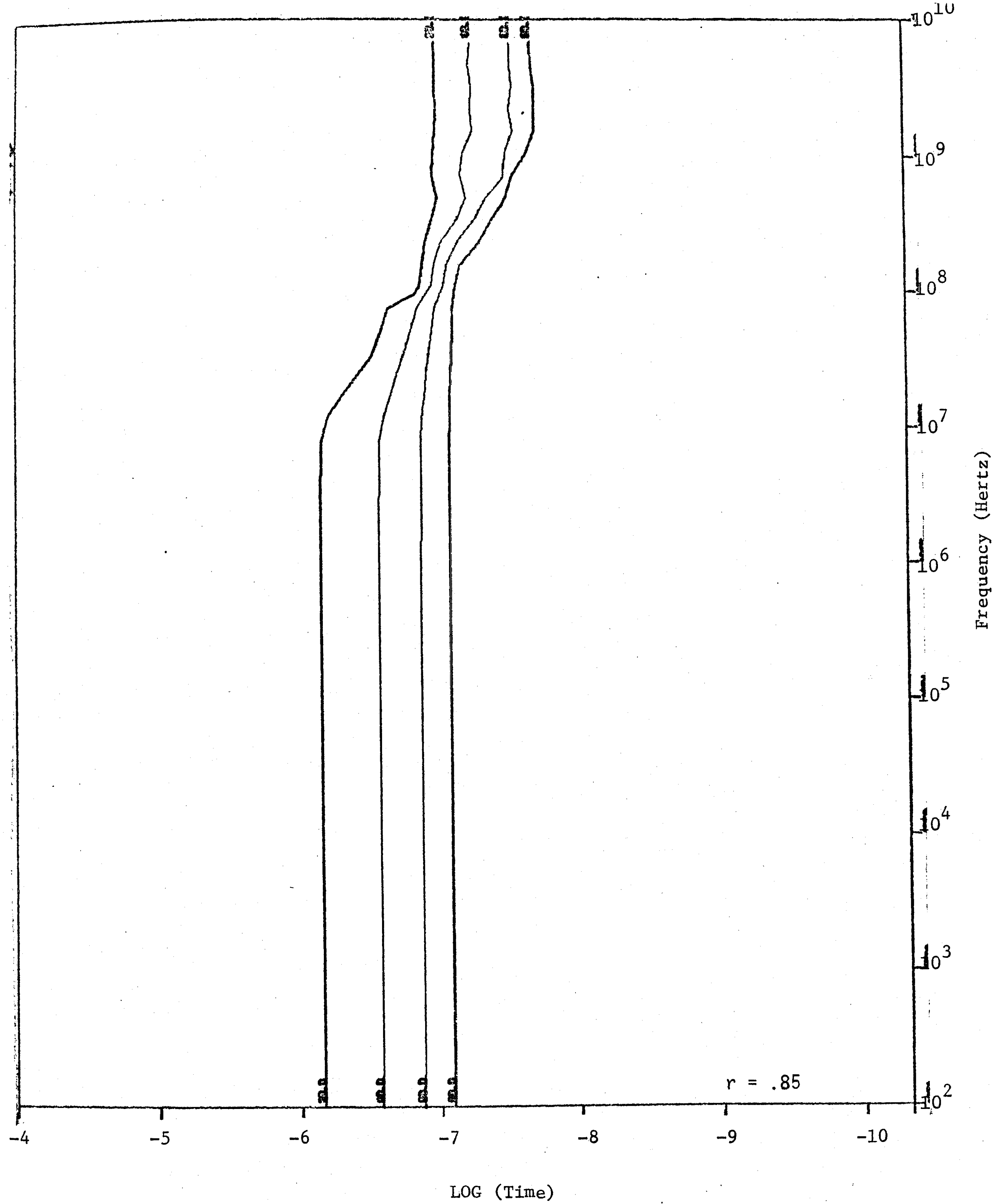
10^{10}
10^9
10^8
10^7
10^6
10^5
10^4
10^3
10^2
Frequency (Hertz)
r = 1.
-4
-5
-6
-7
-8
-9
-10
LOG (Time)

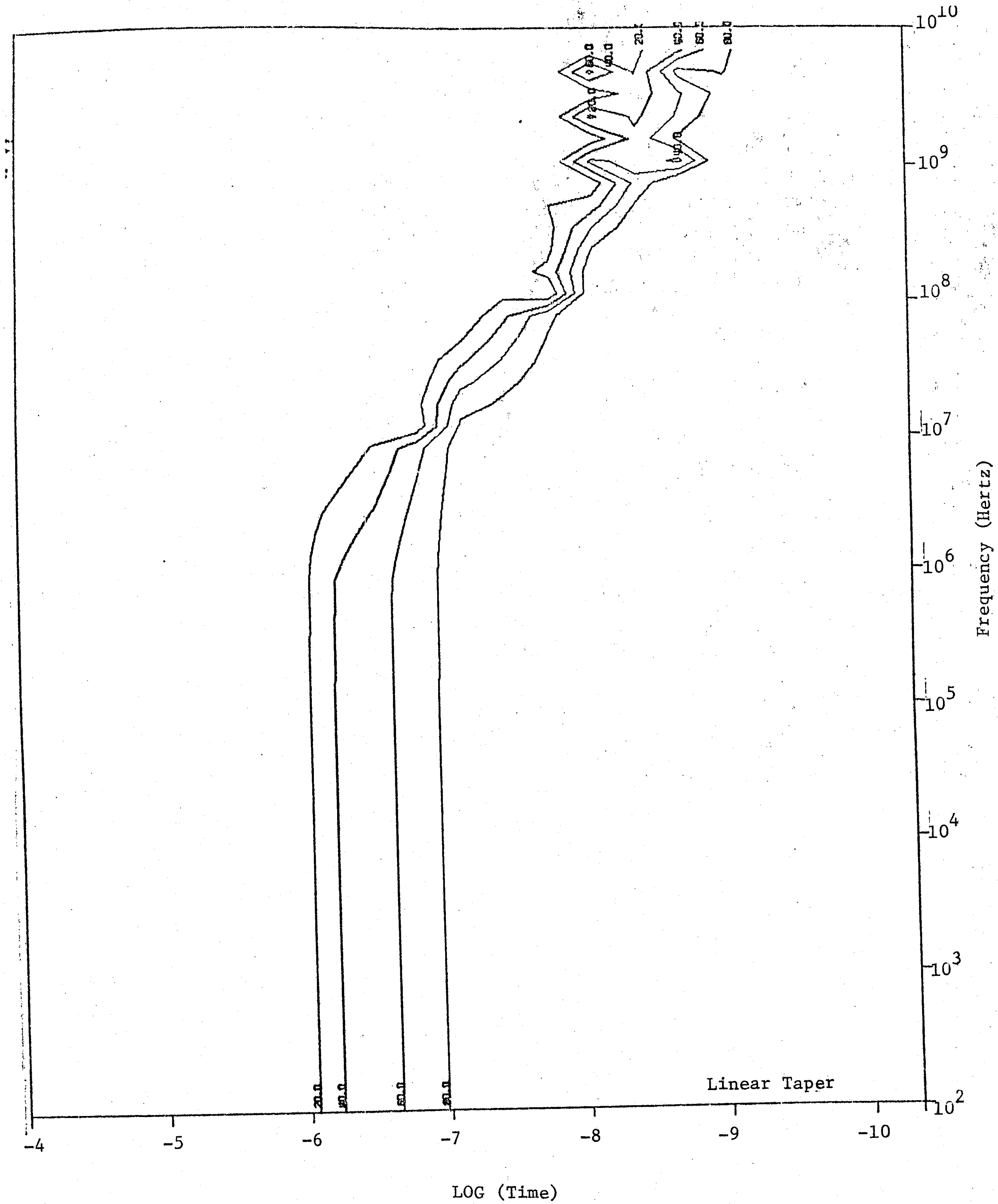


r = .95
LOG (Time)
Frequency (Hertz)
-4
-5
-6
-7
-8
-9
-10
10^{2}
10^{3}
10^{4}
10^{5}
10^{6}
10^{7}
10^{8}
10^{9}
10^{10}



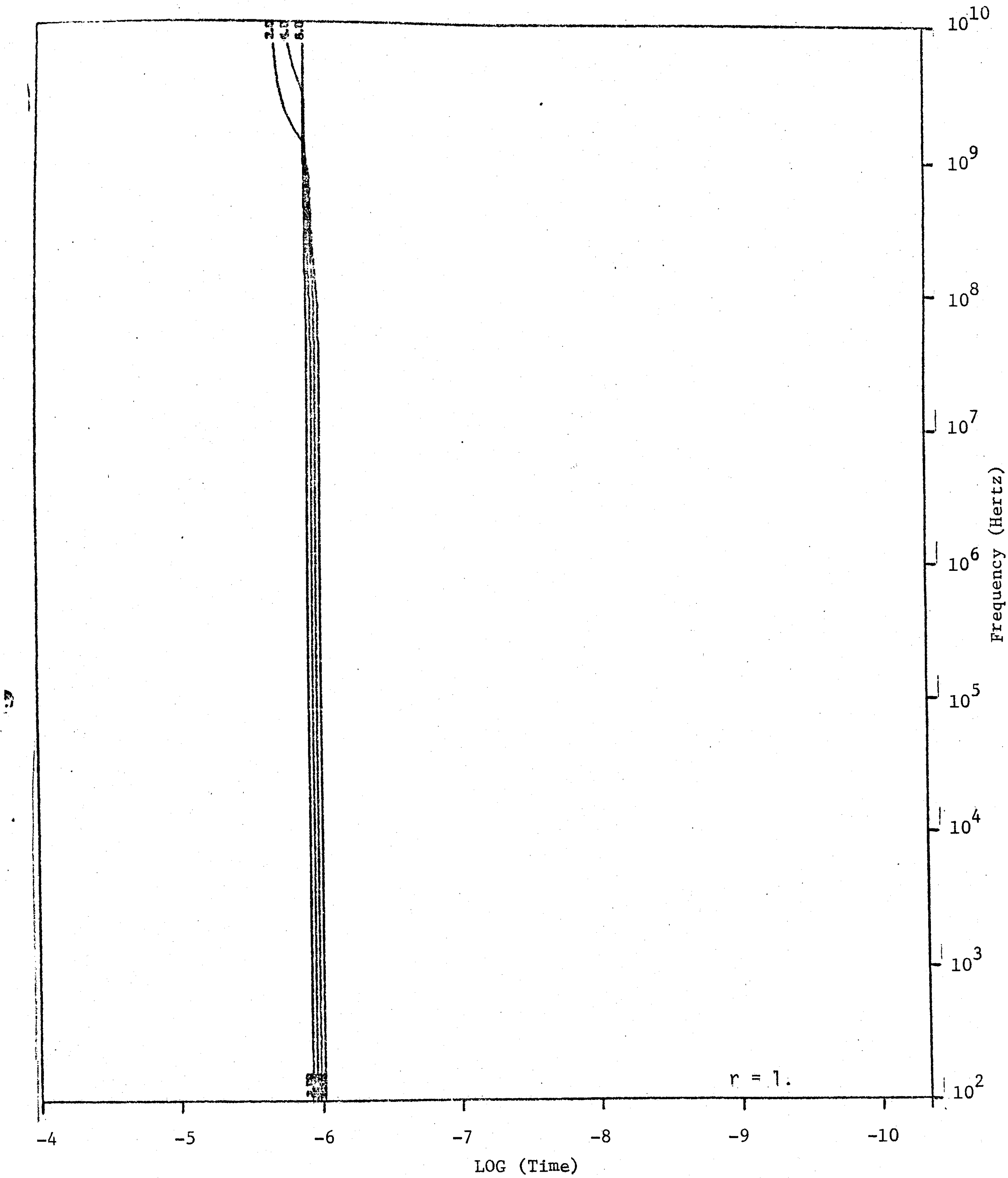
20.0
40.0
60.0
80.0
r = .9
10^{10}
10^9
10^8
10^7
10^6
10^5
10^4
10^3
10^2
Frequency (Hertz)
-4
-5
-6
-7
-8
-9
-10
LOG (Time)


10^{10}
10^9
10^8
10^7
10^6
10^5
10^4
10^3
10^2
Frequency (Hertz)
r = .85
-4
-5
-6
-7
-8
-9
-10
LOG (Time)

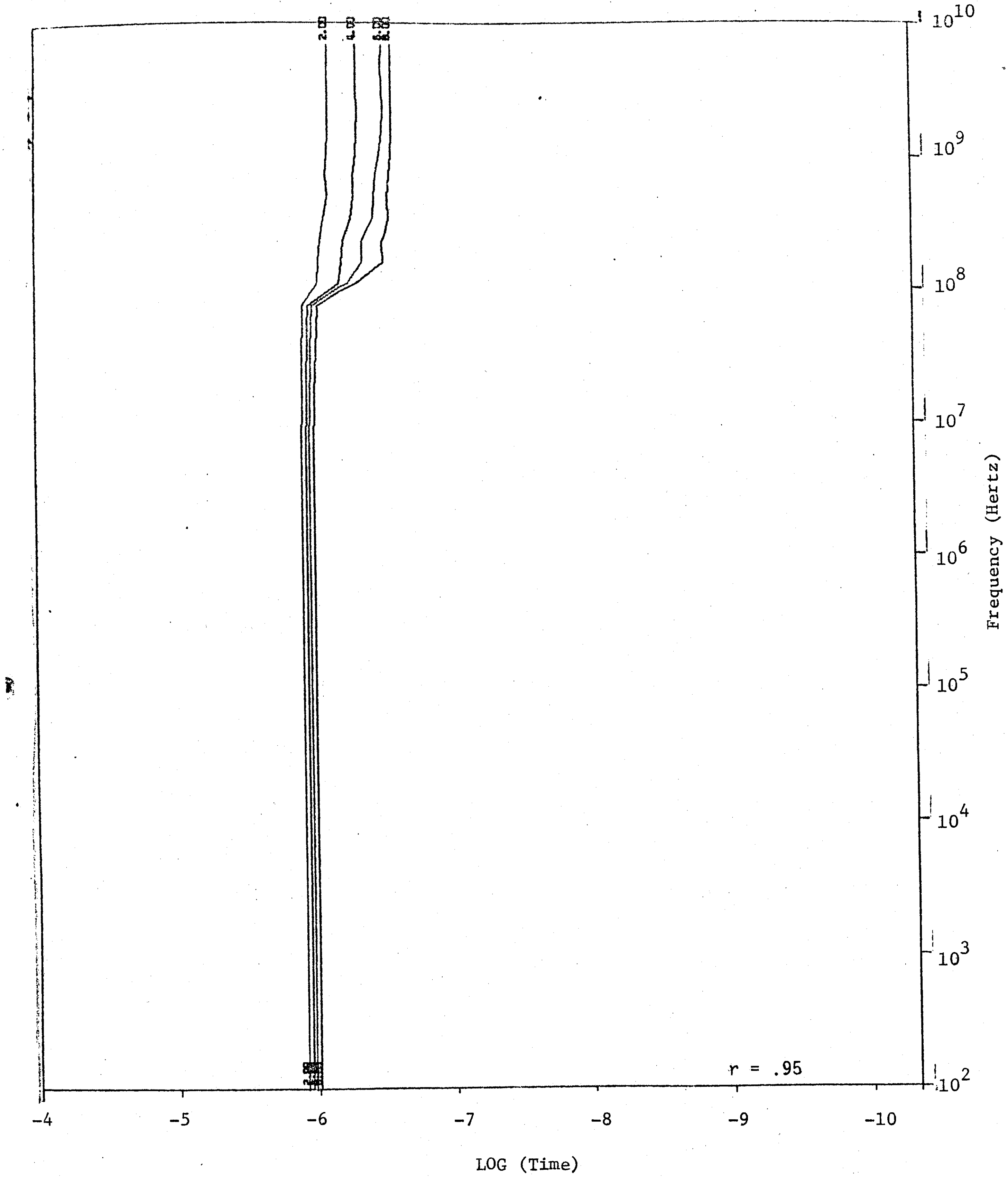

Linear Taper
LOG (Time)
Frequency (Hertz)
-4
-5
-6
-7
-8
-9
-10
10^2
10^3
10^4
10^5
10^6
10^7
10^8
10^9
10^{10}

APPENDIX F

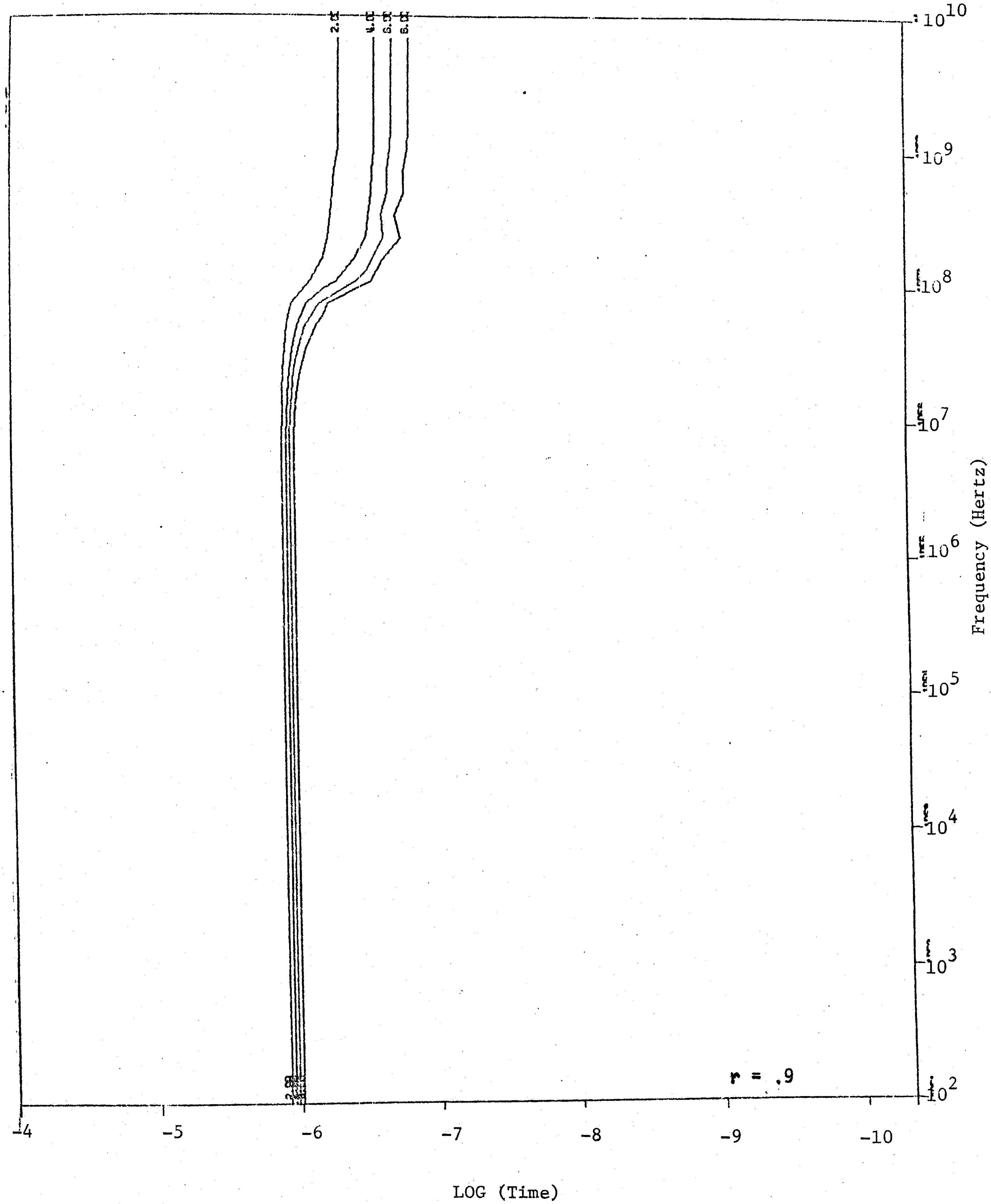
Magnitude Errors of 2% to 8% in Steps of 2%.

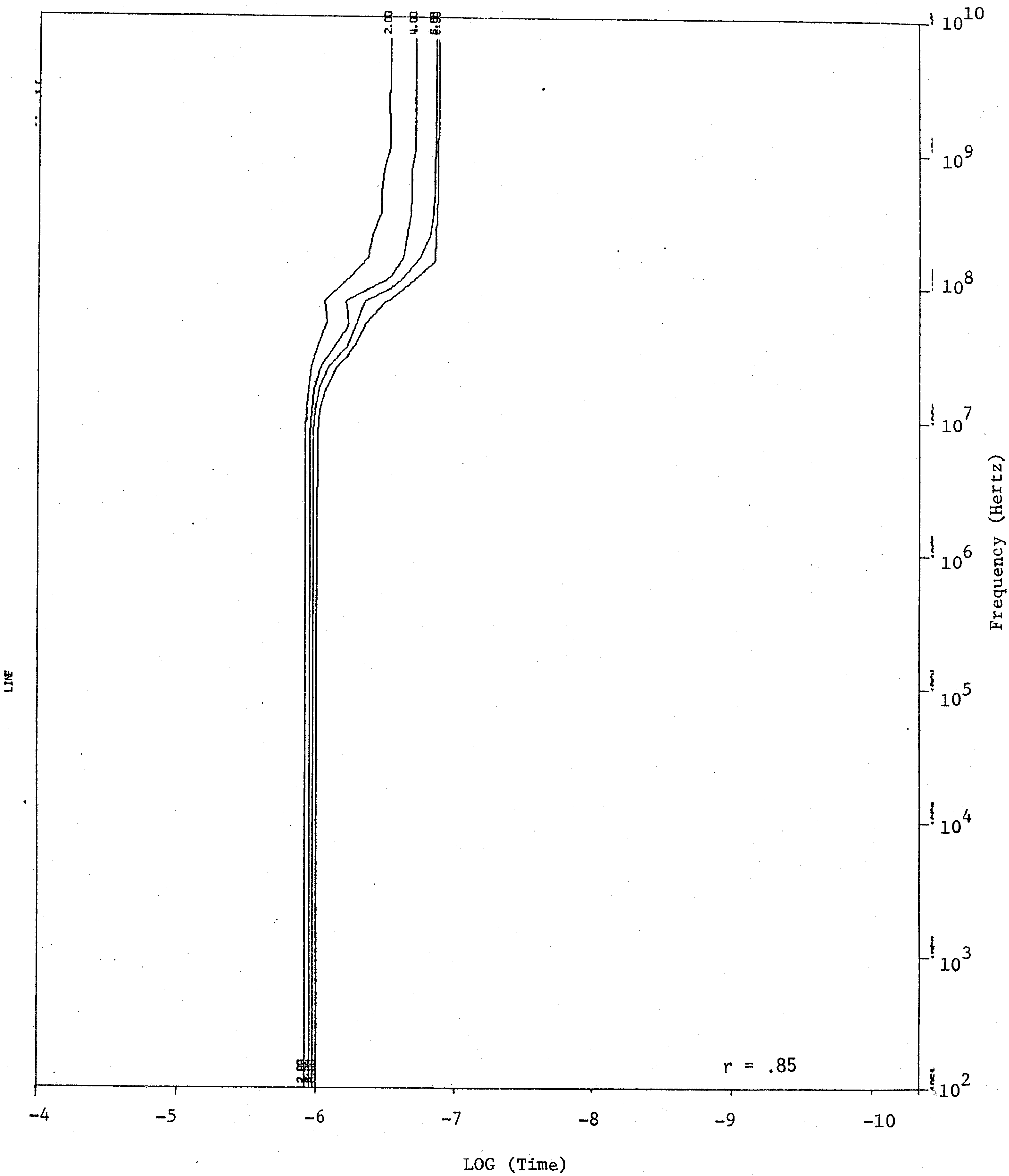


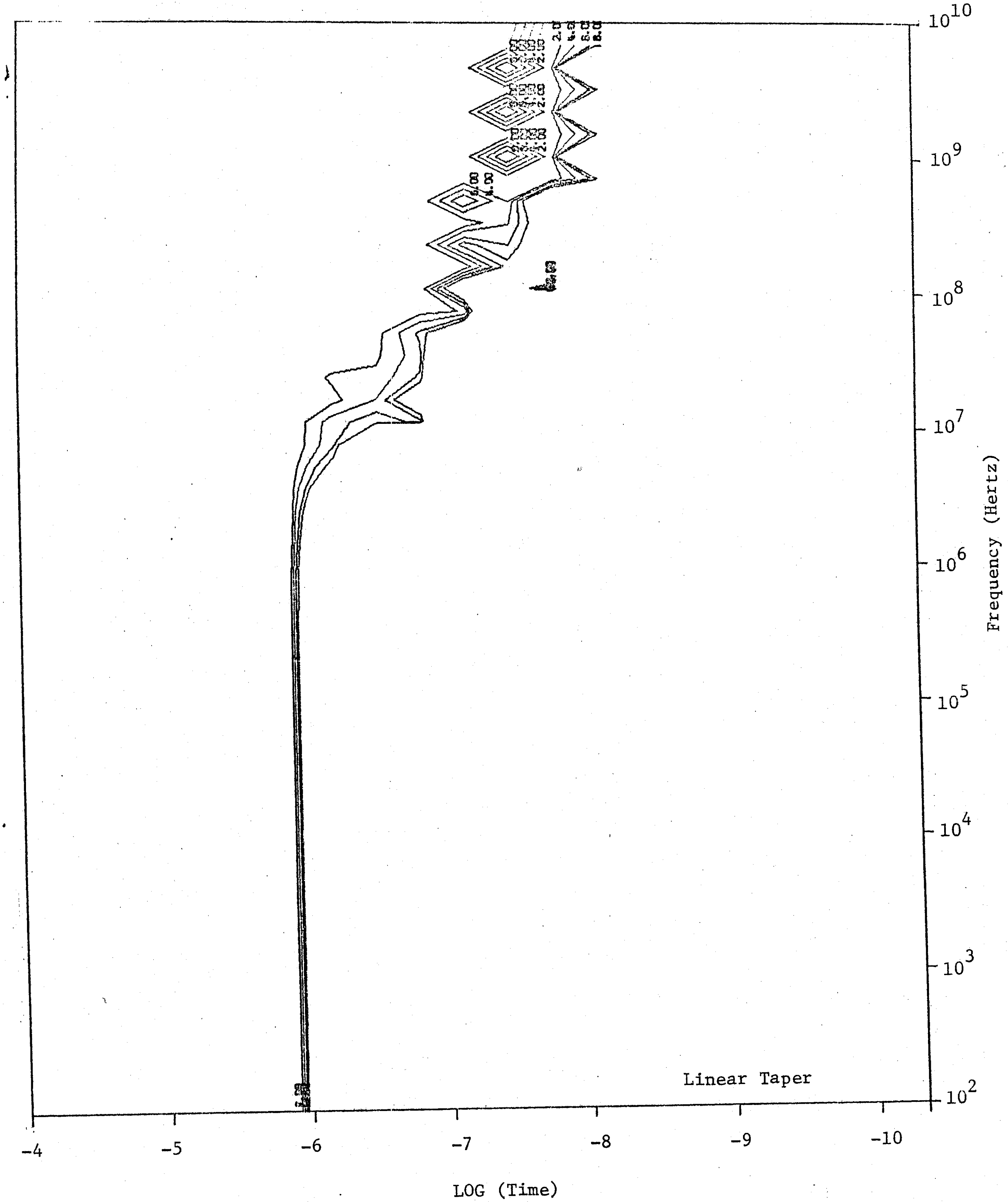
2.5
5.0
8.0
r = 1.
-4
-5
-6
-7
-8
-9
-10
LOG (Time)
10^{10}
10^9
10^8
10^7
10^6
10^5
10^4
10^3
10^2
Frequency (Hertz)



r = .95
-4
-5
-6
-7
-8
-9
-10
LOG (Time)
10^{10}
10^9
10^8
10^7
10^6
10^5
10^4
10^3
10^2
Frequency (Hertz)

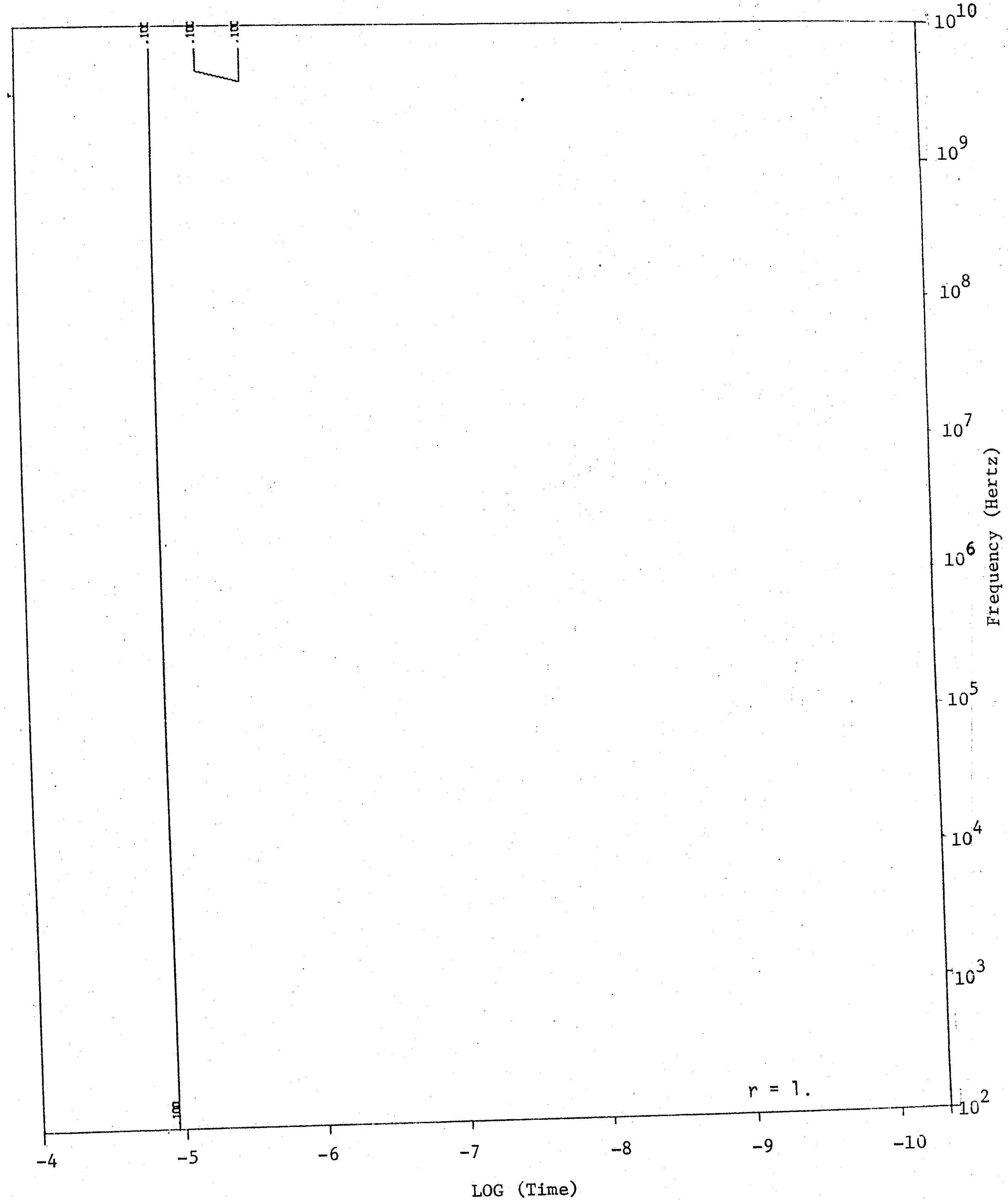

2.0
4.0
6.0
8.0
r = .9
-4
-5
-6
-7
-8
-9
-10
LOG (Time)
10^{2}
10^{3}
10^{4}
10^{5}
10^{6}
10^{7}
10^{8}
10^{9}
10^{10}
Frequency (Hertz)


LINE
2.00
4.00
r = .85
-4
-5
-6
-7
-8
-9
-10
LOG (Time)
10^{10}
10^9
10^8
10^7
10^6
10^5
10^4
10^3
10^2
Frequency (Hertz)

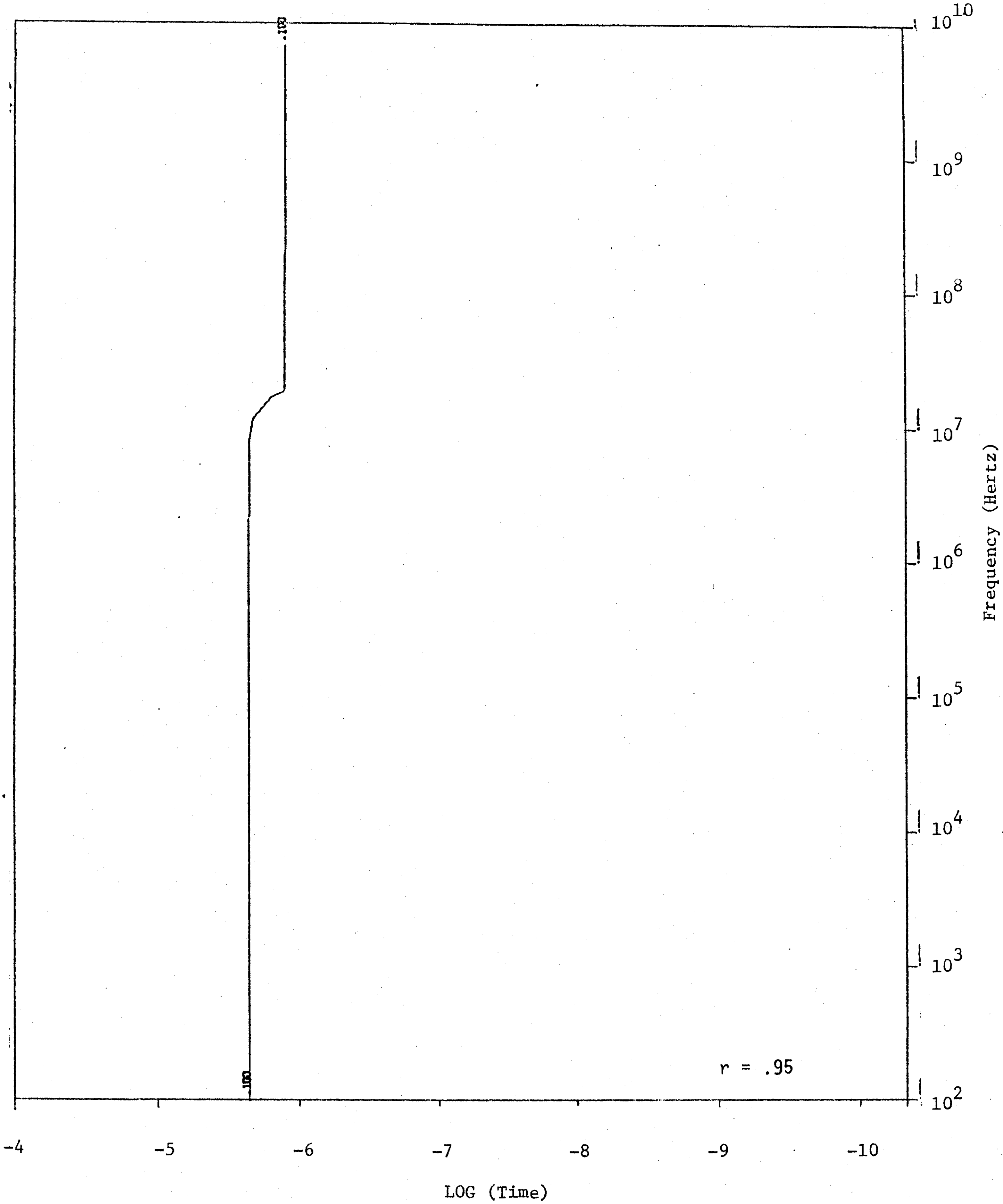

Linear Taper
LOG (Time)
-4
-5
-6
-7
-8
-9
-10
Frequency (Hertz)
10^{2}
10^{3}
10^{4}
10^{5}
10^{6}
10^{7}
10^{8}
10^{9}
10^{10}

APPENDIX G

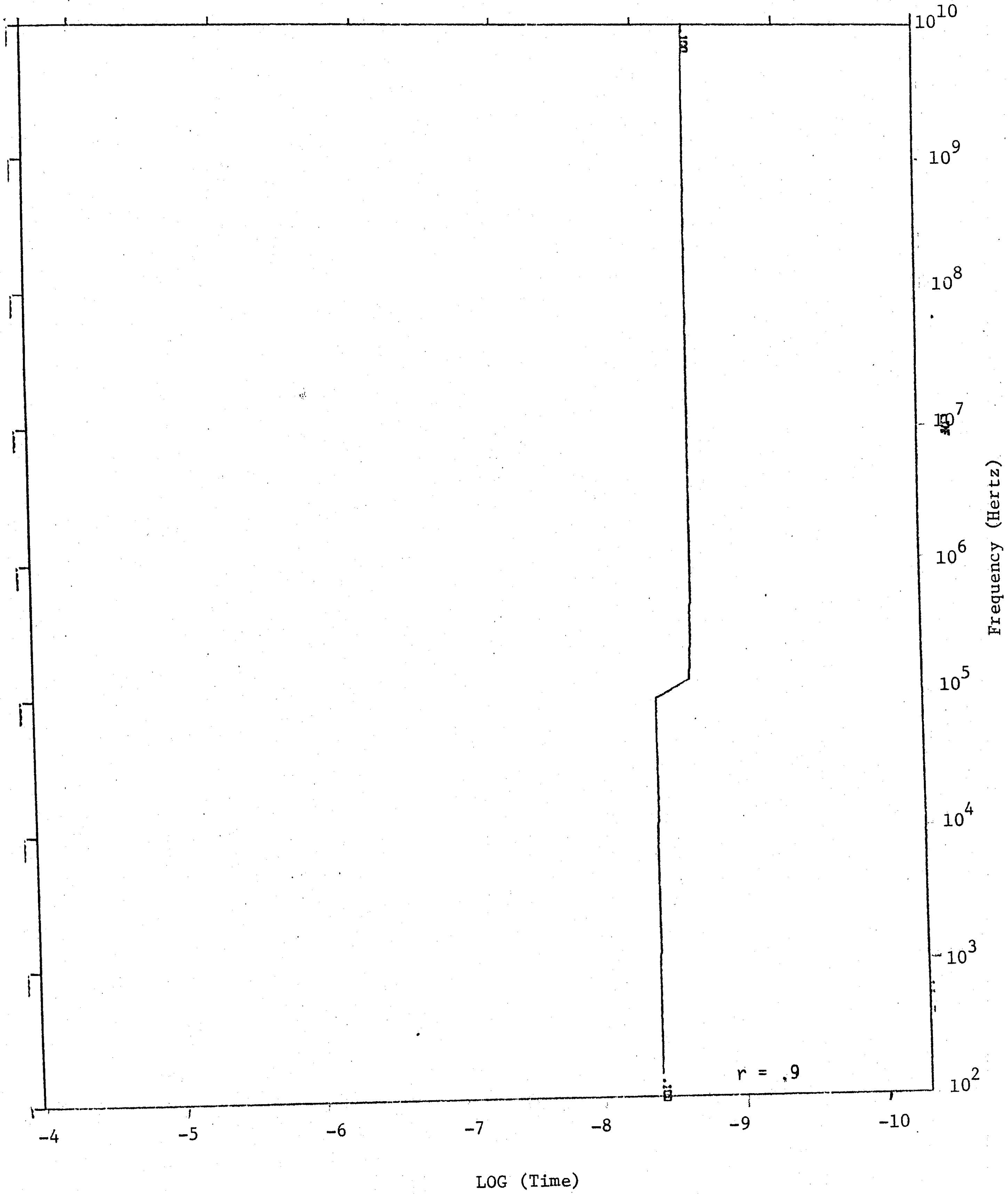
Magnitude Error of .1% .



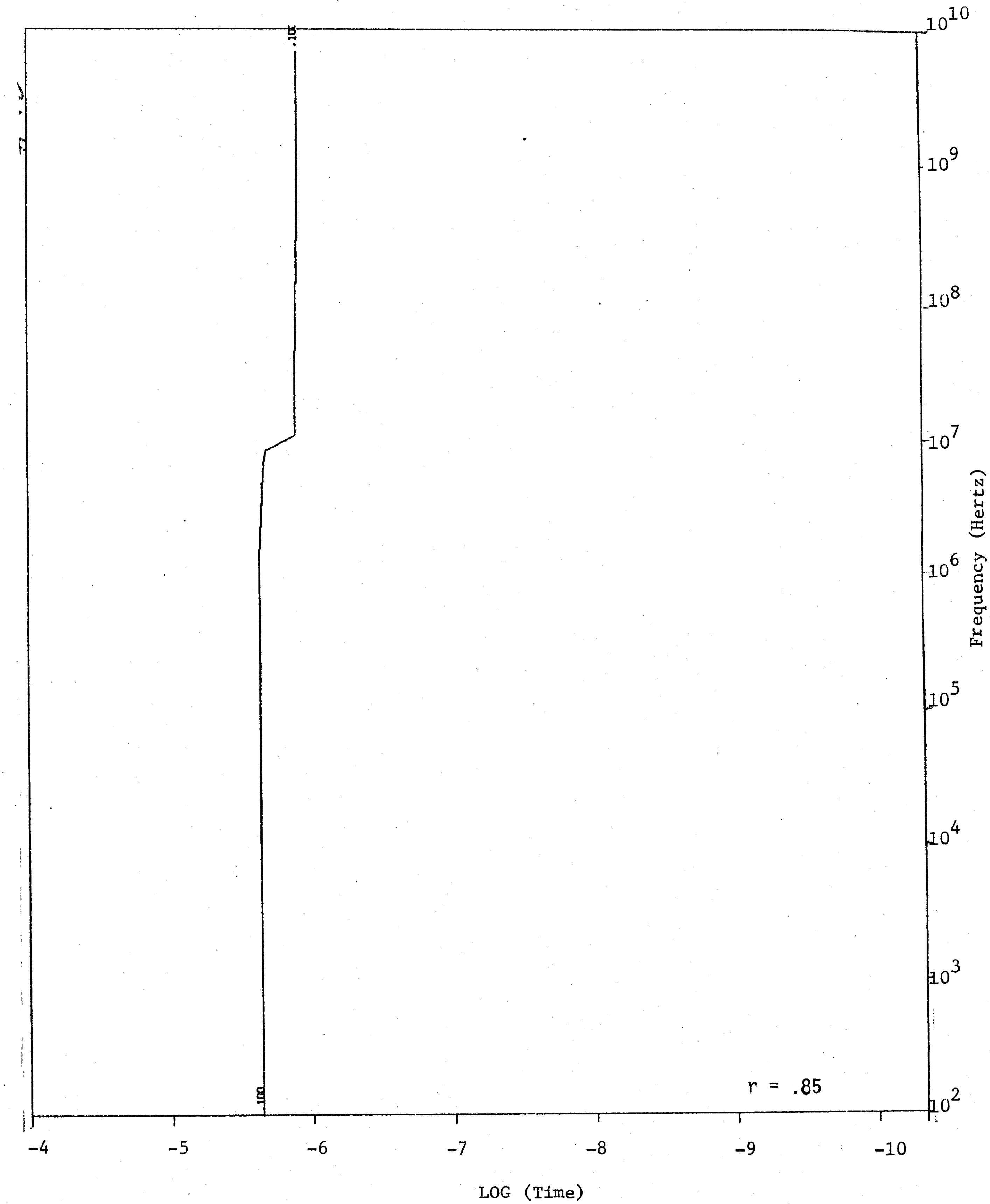
10^10
10^9
10^8
10^7
10^6
10^5
10^4
10^3
10^2
Frequency (Hertz)
-4
-5
-6
-7
-8
-9
-10
LOG (Time)
r = 1.



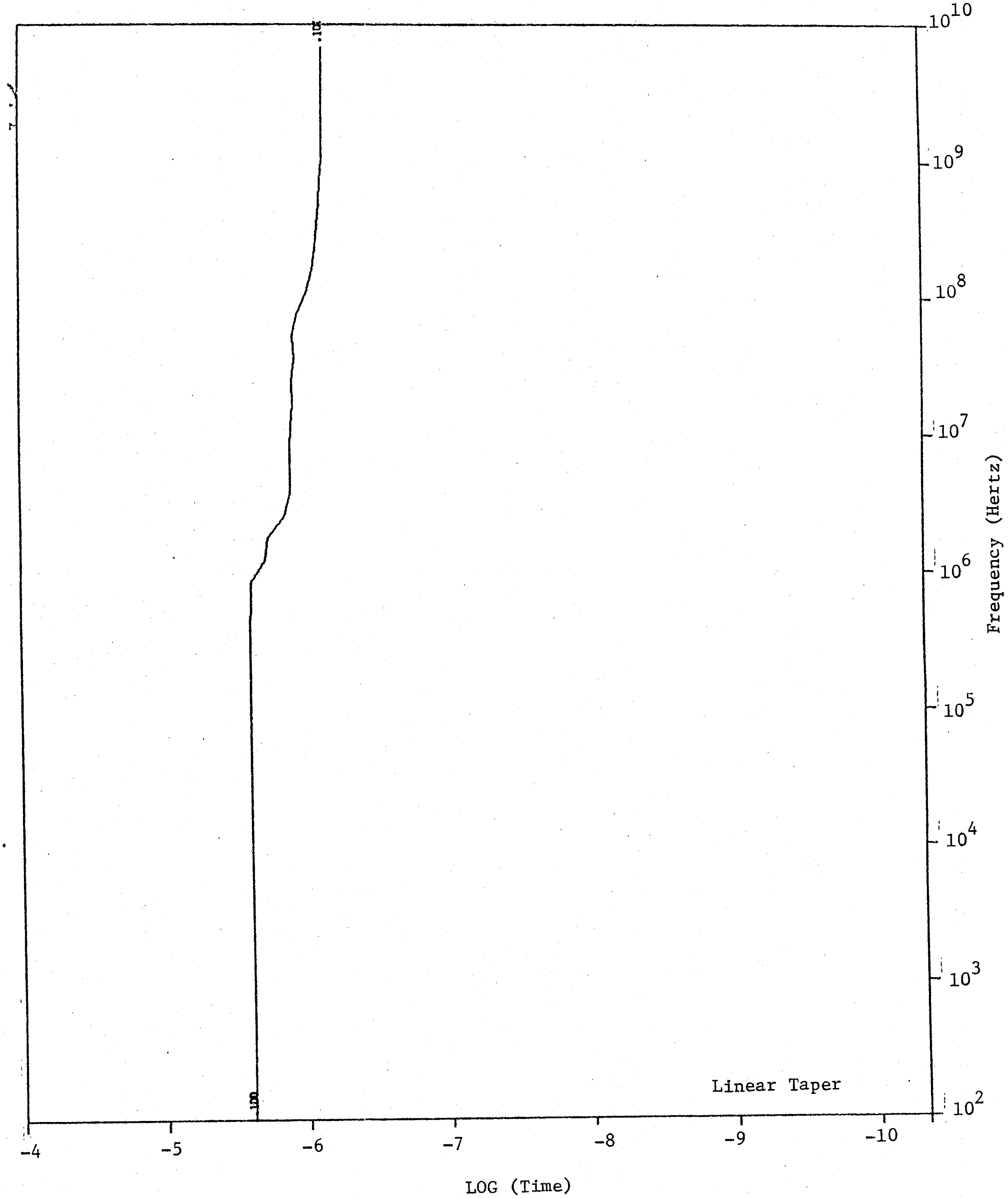
.100
.100
r = .95
10^{10}
10^9
10^8
10^7
10^6
10^5
10^4
10^3
10^2
Frequency (Hertz)
-4
-5
-6
-7
-8
-9
-10
LOG (Time)



10^{10}
10^{9}
10^{8}
10^{7}
10^{6}
10^{5}
10^{4}
10^{3}
10^{2}
Frequency (Hertz)
r = .9
-4
-5
-6
-7
-8
-9
-10
LOG (Time)



10^{10}
10^{9}
10^{8}
10^{7}
10^{6}
10^{5}
10^{4}
10^{3}
10^{2}
Frequency (Hertz)
r = .85
100
-4
-5
-6
-7
-8
-9
-10
LOG (Time)



.105
100
Linear Taper
10^{10}
10^9
10^8
10^7
10^6
10^5
10^4
10^3
10^2
Frequency (Hertz)
-4
-5
-6
-7
-8
-9
-10
LOG (Time)