

Mathematics Notes

Note 79 - Vol. III

17 August 1981

Investigation of Procedures for Automatic Resonance  
Extraction From Noisy Transient Electromagnetics Data

Volume III - Translation of Prony's Original Paper  
and Bibliography of Prony's Method

Jon R. Auton  
Michael L. Van Blaricum

General Research Corporation  
5383 Hollister Avenue  
Santa Barbara, CA 93111

Abstract

This document contains a bibliography of reports, papers and books that deal directly with Prony's method. Section 2.0 of this volume contains a translation from the French of Prony's original paper.

Acknowledgment

This work sponsored by the Office of Naval Research under Contract No. N00014-80-C-0299.

VOLUME III  
TABLE OF CONTENTS

<u>Section</u>		<u>Page</u>
1.0	INTRODUCTION - - - - -	4
2.0	TRANSLATION OF PRONY'S ORIGINAL PAPER - - - - -	5
3.0	JOURNAL ARTICLES, COMPANY AND GOVERNMENT REPORTS AND THESES - - - - -	21
4.0	PAPERS PRESENTED AT CONFERENCES, SYMPOSIA AND MEETINGS	71
5.0	BOOKS - - - - -	76

## 1.0 INTRODUCTION

This document contains a bibliography of reports, papers and books that deal directly with Prony's method. No attempt has been made to include papers which fall under the general categories of approximation with exponential sums, linear prediction, parameter estimation or system identification even though these topics are closely related to Prony's method. The papers which we found useful in these fields are referenced in the other volumes which make up this report.

Section 2.0 of this volume contains a translation from the French of Prony's original paper. This has been included for two reasons. The first being that while Prony's original paper is highly referenced, we suspect that most people have never seen it let alone have read it. The second reason for including it is that we find it extremely interesting to read.

Section 3.0 contains an annotated bibliography of journal articles, reports and theses that deal with Prony's method. In most cases, the annotation was taken from the author's abstract of his paper. Section 4.0 contains lists of papers presented at conferences, symposia and meetings. Section 5.0 contains a list of books dealing with the subject of Prony's method.

It is hoped that the users of this list will cross reference it with their own bibliographies on Prony's method and will let ETI know of any additions which they find.

2.0 TRANSLATION OF PRONY'S ORIGINAL PAPER

EXPERIMENTAL AND ANALYTICAL ESSAY  
ON  
THE EXPANSION PROPERTIES OF ELASTIC FLUIDS  
AND ON THE FORCE OF EXPANSION OF  
WATER VAPOR AND ALCOHOL VAPOR AT DIFFERENT  
TEMPERATURES

By R. Prony, 1795  
Translated by Dr. Ann Sanders/ETI

GENERAL CONSIDERATIONS

In approximately the last 40 years, Physics has been enriched greatly by a large number of observations made with extreme care by clever and experienced men. This knowledge is increasing daily, and its substance is becoming ever more precious as the perfection of new instruments provides more and more precision to the experiments; phenomena have been correlated, compared and classified; the language of an important part of science has become analytical; reasonable theories have replaced futile and often absurd theories, which had been taught until the middle of this century.

The study of nature in terms of its observation and the knowledge of its operation, seems to me to consist of two aspects which should not be confused: the explanation of effects and their measurement.

The explanation of effects consists of findings, in a class of multifaceted phenomena, those simple or basic phenomena from which the others are derived, or of which they consist in various combinations. It also consists of showing how, from the most varied appearance, one can untangle the operation and the being of the elements considered to be the basis of the system.

Thus, by starting from the affinity of certain substances which are considered as basic phenomena, it has been found that meteorological phenomena, combustion phenomena etc., result from these same affinities, and show up under different disguises which, until recently, hid their nature from physicists. It is by this decomposition of complex effects into basic effects that we can discover certain secrets of nature, which, while it allows us to lift one corner of the veil which covers it, keeps another corner tied by a knot which our hand cannot untie.

The measurement of effects is the evaluation of the degree to which each effect is variable when one varies either the causes which produce it, or other effects to which it is tied. It is known, for instance, that the tendency of fluids to vaporize is on the one hand enhanced by temperature, and on the other hand reduced by the atmospheric pressure, and that vaporization occurs only when the former force exceeds the latter one; in addition to this fact, it may be necessary to give the values of the pressures which produce vaporization equilibrium at different temperatures. This same kind of reasoning is applicable to infinitely many other phenomena.

It can thus be seen that the explanation of effects, whose great advantage is to simplify science and to correlate its various aspects, by analysis and by decomposition into basic phenomena, must be complemented by the measurement of these same effects, which is always very useful and often indispensable when it is desired to apply theoretical discoveries to the needs of society.

Only experiments can provide the initial data on the measurement of physical effects; following those, however, calculations can be very useful, either to obtain results intermediate to those obtained from the experiments, or to correct experimental anomalies. The method used in that case is called interpolation. Its aim is to find a relationship between two or three variables, such that when a certain value is given to one or two of the variables, a equally specific value is obtained for the second or third variable when the problem is considered in that manner,

it can be solved in an infinite number of ways, since there are an infinite number of functions which can be obtained by the same substitutions. It would be erroneous, however, to think that all those solutions are equally applicable to a given case. Even though nature is governed by seemingly few and simple general laws, it has as many modifications in its workings as variety in its forms, and each measurable phenomenon is always correlated to a specific function which represents it exclusively.

The problem of interpolation thus has two distinct parts; the first part consists of satisfying given numbers, and the second part consists of finding the function applicable to the phenomenon amongst all the functions which satisfy the conditions.

In #19 of my Analysis Lessons, I have given a solution to the first part of the problem which is used very often, especially as a method of correction. On the same subject, Lagrange has published a beautiful treatise\* where he handled the subject in a more general way than had been done before. The students which know the theory explained in my lessons #19, 20 and 21, will be able to study this work with ease, and will be able to gain much from the time they will have spent on it.

From the knowledge we have now, the solution to the second part does not seem to be governed by general rules, especially when the number of observations is small, and when the observations do not span a wide range. The only guidelines available in this painful search are careful attention to all the details and the progression of the experiments, repetition of measurements, and analogies. These difficulties, combined with the difficulty of precision in experiments, are the reason why exact determinations of physical laws are very rare in Physics.

In 1790, I had the opportunity to follow very detailed and careful experiments on the expansion force of water vapor, and it was my aim to find the applicable formula. The regularity of the data led me to believe that the task would be easier than it actually was. However, after some effort, I found a certain function which not only perfectly expressed the relationship between the temperature and the spring of the

---

\* Footnote illegible.

watery vapor, but which seemed to apply in general to phenomena depending on elastic fluids. I applied them to careful experiments which Prieur made on the expansion properties of air and of different air-form fluids; this strengthened my opinion, and I decided to publish my results.

The first observation which directed me to the true form of the function was the consideration of certain geometric progressions which are present in certain phenomena related to elastic fluids. One of the most remarkable examples of this is the relationship between the density of atmospheric layers and their respective height. Since this relationship is exponential, I suspected that in other cases, where one such quantity would not be sufficient, two or more could be introduced. This led to an equation of the form  $\tau = \mu_1 \rho_1^x + \mu_2 \rho_2^x + \mu_3 \rho_3^x + \dots + \mu_n \rho_n^x$  where  $\tau$  and  $n$  are the two variables and  $\mu_1, \mu_2, \dots, \rho_1, \rho_2, \dots$  are the constants appropriate to the given phenomenon.

It is known that the above equation results from an equation with linear finite differences or gives the general term in a recurrent series of order  $n$ . However, series of the type where a given term is obtainable from a certain number of preceding terms seems indeed to be compatible with natural effects where elasticity plays a large part. The conservation of living forces which is contained in this property of bodies always causes the actual state to be dependent on the previous states. The research of Lagrange, which I mentioned above, is also based on recurrent series. He has given several methods to find those which must interpolate a given series, where one can observe the elegance and the depth which one comes to expect from such a great analyst. Since the method I have used in my calculations differs from his methods, which I did not know when I started this work, I will now describe it.



## FIRST PART

### Interpolation Method Applicable to Phenomena Depending on Elastic Fluids

The experiments must be directed as much as possible towards yielding equidistant results; when this condition cannot be met (which must happen very rarely), but when the results are nevertheless numerous and close enough together, they can be made to be equidistant. This can be done either by graphic methods, by drawing the experimental curve, or by calculations, namely by considering three consecutive results  $Z_i$ ,  $Z_{ii}$  and  $Z_{iii}$ , where the second and the third are a distance  $x^i$  and  $x^{ii}$  respectively from the first. The result  $Z$ , which belongs to the series of equidistant points, and which is a distance  $x$  from  $Z_i$ , can be calculated from the following equation, which is obtained from those in my 19<sup>th</sup> lesson of analysis:

$$Z = \frac{x^{ii} - x}{x^i} \cdot \frac{x^i - x}{x^{ii}} Z_i + \frac{x}{x^{ii} - x^i} \left( \frac{x^{ii} - x}{x^i} Z_{ii} - \frac{x^i - x}{x^{ii}} Z_{iii} \right)$$

This formula can be simplified greatly by calculating the difference between  $Z$  and  $Z^i$ . By setting  $Z_{ii} - Z_i = \omega^i$  and  $Z_{iii} - Z_i = \omega^{ii}$ , one obtains

$$Z - Z_i = \frac{x}{x^{ii} - x^i} \left( \frac{x^{ii} - x}{x^i} \omega^i - \frac{x^i - x}{x^{ii}} \omega^{ii} \right).$$

In order to avoid mistakes, it is advisable to apply both the formula and the graphic method, which, when applied carefully and on a large scale, will give a precision comparable to the experiments themselves.

Once this preparation has been made (and as I said, the cases where this is necessary will be extremely rare), a certain number of equidistant results will be taken which will encompass either the entire experimental range, or a large part thereof. Then the variable  $Z$  is taken which indicates the measurement of successive effects which correspond to certain values of another variable  $x$ , which indicates to which term on a given scale the effects  $Z$  approach.

To satisfy a number  $2n$  of results, one will generally have

$$Z = \mu_i \rho_i^x + \mu_{ii} \rho_{ii}^x + \mu_{iii} \rho_{iii}^x + \dots + \mu_{(n)} \rho_{(n)}^x \dots \dots \dots \quad (1)$$

and to satisfy a number  $2n + 1$  of results,

$$Z = \mu_i \rho_i^x + \mu_{ii} \rho_{ii}^x + \dots + \mu_{(n)} \rho_{(n)}^x + \mu_{(n+1)} \dots \dots \dots \quad (2)$$

where  $\mu_i, \mu_{ii}, \mu_{iii},$  etc., and  $\rho_i, \rho_{ii}, \rho_{iii},$  etc., are constants whose values are determined from the experimental results, as will soon be seen.

I have given two general formulas, although usually one of the two would be sufficient, but I had a simplification in mind which is an important inclusion in the method, and which will now be described. The determination of  $\rho_i, \rho_{ii}, \rho_{iii},$  etc., depends on the solution of an equation, and by using the second formula, one satisfies an odd number of observations by an equation which is of the same order as that required for satisfying the next smaller even number of observations. Thus, one can satisfy four or five observations by calculating a second order equation, six or seven observations by calculating a third order equation, eight or nine observations by calculating a fourth order equation, and so on. It will almost never happen that eight or nine results must be included in the formula, and without exceeding the limits where methods for the solution of numerical equations exist, one will be able to treat all the cases which Physics ordinarily presents. In addition to that advantage, it should be mentioned that there will be at most only half as many values of  $Z$  as there are observation points, instead of with the formulas which refer to parabolic curves, and which always have as many terms as there are observation points.

Here then is the method by which the constants in equations (1) and (2) can be determined, using the given results.

First case, where the number of observations is even:

In my analysis lessons numbers 20 and 21, I have shown that equation (1) produced the general term in a recurring series of order n, and also that the terms in such a series, taken at certain equal intervals, always produced series of the same order. Given that, let there be the following two series, the first of which produces the observed results of the particular values of Z furnished by the experiment and the second of which consists of the corresponding values of x, namely the observed results

$$Z_0; Z_i; Z_{ii} \dots Z_{(n)}; Z_{(n+1)}; Z_{(n+2)} \dots Z_{2n-1}$$

and the corresponding values of x

$$0; x_i; 2x_i \dots nx_i; (n+1)x_i; (n+2)x_i \dots (2n-1)x_i.$$

The quantities  $Z_0, Z_i, Z_{ii}$ , etc. must form a recurring series whose order must be found.

Let  $A_0, A_i, A_{ii}$  be undetermined coefficients, satisfying the relationships

$$\begin{aligned} A_0 Z_0 + A_i Z_i + A_{ii} Z_{ii} + \dots + A_{(n)} Z_{(n)} &= 0 \\ A_0 Z_i + A_i Z_{ii} + A_{ii} Z_{iii} + \dots + A_{(n)} Z_{(n+1)} &= 0 \\ A_0 Z_{ii} + A_i Z_{iii} + A_{ii} Z_{iv} + \dots + A_{(n)} Z_{(n+2)} &= 0 \\ A_0 Z_{iii} + A_i Z_{iv} + A_{ii} Z_v + \dots + A_{(n)} Z_{(n+3)} &= 0 \\ \dots & \\ \dots & \\ A_0 Z_{(n-1)} + A_i Z_{(n)} + A_{ii} Z_{(n+1)} + \dots + A_{(n)} Z_{(2n-1)} &= 0. \end{aligned}$$

(For convenience,  $A_{(n)}$  may be set to 1 in the numerical applications.)

These n equations will produce n ratios  $\frac{A_0}{A_{(n)}} ; \frac{A_i}{A_{(n)}} ; \frac{A_{ii}}{A_{(n)}} \dots \frac{A_{(n-1)}}{A_{(n)}}$

which make up the desired order of the series, and one will obtain:

For n = 1

$$A_0 : A_i = -Z_i : Z_0$$

Value obtained from two observations.

For n = 2

$$A_0 : A_{ii} = \frac{Z_i Z_{iii} - Z_{ii} Z_{ii}}{Z_0 Z_{ii} - Z_i Z_i}$$

$$A_i : A_{ii} = \frac{Z_i Z_{ii} - Z_0 Z_{iii}}{Z_0 Z_{ii} - Z_i Z_i}$$

Values obtained from four observations.

For n = 3

$$A_0 : A_{iii} = \frac{+(Z_{iii} Z_{iii} - Z_{ii} Z_{iv}) Z_{iii} + (Z_i Z_{iv} + Z_{ii} Z_{iii}) Z_{iv} + (Z_{ii} Z_{ii} - Z_i Z_{iii}) Z_v}{-(Z_{iii} Z_{iii} - Z_{ii} Z_{iv}) Z_0 - (Z_i Z_{iv} - Z_{ii} Z_{iii}) Z_i - (Z_{ii} Z_{ii} - Z_i Z_{iii}) Z_{ii}}$$

$$A_i : A_{iii} = \frac{+(Z_i Z_{iv} - Z_{ii} Z_{iii}) Z_{iii} + (Z_{ii} Z_{ii} - Z_0 Z_{iv}) Z_{iv} + (Z_0 Z_{iii} - Z_i Z_{ii}) Z_v}{-(Z_i Z_{iv} - Z_{ii} Z_{iii}) Z_i - (Z_{ii} Z_{ii} - Z_0 Z_{iv}) Z_{ii} - (Z_0 Z_{iii} - Z_i Z_{ii}) Z_{iv}}$$

$$A_{ii} : A_{iii} = \frac{+(Z_{ii} Z_{ii} - Z_i Z_{iii}) Z_{iii} + (Z_0 Z_{iii} - Z_i Z_{ii}) Z_{iv} + (Z_i Z_i - Z_0 Z_{ii}) Z_v}{-(Z_{ii} Z_{ii} - Z_i Z_{iii}) Z_{ii} - (Z_0 Z_{iii} - Z_i Z_{ii}) Z_{iii} - (Z_i Z_i - Z_0 Z_{ii}) Z_{iv}}$$

Values obtained from six observations.

etc.

etc.

etc.

Then, upon solving the equation,

$$A_0 + A_i x + A_{ii} x^2 + A_{iii} x^3 + \dots + A_{(n)} x^n = 0,$$

the n roots which will be obtained will be the n values  $\rho_i^{x_i}, \rho_{ii}^{x_i}, \rho_{iii}^{x_i}, \dots, \rho_{(n)}^{x_i}$  from which the values of  $\rho_i, \rho_{ii}, \rho_{iii},$  etc. will be obtained. Finally, the quantities  $\mu_i, \mu_{ii}, \mu_{iii},$  etc. will be given by the equations:

$$\mu_i = \frac{(Z - \rho_{ii}^{x_i}) (Z - \rho_{iii}^{x_i}) (Z - \rho_{iv}^{x_i}) \dots (Z - \rho_{(n)}^{x_i})}{(\rho_i^{x_i} - \rho_{ii}^{x_i}) (\rho_i^{x_i} - \rho_{iii}^{x_i}) (\rho_i^{x_i} - \rho_{iv}^{x_i}) \dots (\rho_i^{x_i} - \rho_{(n)}^{x_i})}$$

$$\mu_{ii} = \frac{(Z - \rho_i^{x_i}) (Z - \rho_{iii}^{x_i}) (Z - \rho_{iv}^{x_i}) \dots (Z - \rho_{(n)}^{x_i})}{(\rho_{ii}^{x_i} - \rho_i^{x_i}) (\rho_{ii}^{x_i} - \rho_{iii}^{x_i}) (\rho_{ii}^{x_i} - \rho_{iv}^{x_i}) \dots (\rho_{ii}^{x_i} - \rho_{(n)}^{x_i})}$$

$$\mu_{iii} = \frac{(Z - \rho_i^{x_i}) (Z - \rho_{ii}^{x_i}) (Z - \rho_{iv}^{x_i}) \dots (Z - \rho_{(n)}^{x_i})}{(\rho_{iii}^{x_i} - \rho_i^{x_i}) (\rho_{iii}^{x_i} - \rho_{ii}^{x_i}) (\rho_{iii}^{x_i} - \rho_{iv}^{x_i}) \dots (\rho_{iii}^{x_i} - \rho_{(n)}^{x_i})}$$

.....

$$\mu_{(n)} = \frac{(Z - \rho_i^{x_i}) (Z - \rho_{ii}^{x_i}) (Z - \rho_{iii}^{x_i}) \dots (Z - \rho_{(n-1)}^{x_i})}{(\rho_{(n)}^{x_i} - \rho_i^{x_i}) (\rho_{(n)}^{x_i} - \rho_{ii}^{x_i}) (\rho_{(n)}^{x_i} - \rho_{iii}^{x_i}) \dots (\rho_{(n)}^{x_i} - \rho_{(n-1)}^{x_i})}$$

by noting that in the numerators all the coefficients of the powers of Z must be changed to subscripts of the same order, i.e., that  $Z^0$  must be replaced by  $Z_0$  (or that all the terms which do not contain Z must be multiplied by  $Z_0$ ), that Z must be replaced by  $Z_1$ , that  $Z^2$  must be replaced by  $Z_{ii}$ , etc., thus one will have in the case of

$$n = 1 \dots \mu_i = Z_0$$

To satisfy two observations.

$$n = 2 \dots \left\{ \begin{array}{l} \mu_i = \frac{Z_i - \rho_{ii}^x Z_0}{\rho_i^x - \rho_{ii}^x} \\ \mu_{ii} = \frac{Z_{ii} - \rho_{ii}^x Z_0}{\rho_{ii}^x - \rho_{ii}^x} \end{array} \right.$$

To satisfy four observations.

$$n = 3 \dots \left\{ \begin{array}{l} \mu_i = \frac{Z_i - (\rho_{ii}^x + \rho_{iii}^x) Z_i + \rho_{ii}^x \rho_{iii}^x Z_0}{(\rho_i^x - \rho_{ii}^x) (\rho_i^x - \rho_{iii}^x)} \\ \mu_{ii} = \frac{Z_{ii} - (\rho_{ii}^x + \rho_{iii}^x) Z_i + \rho_{ii}^x \rho_{iii}^x Z_0}{(\rho_{ii}^x - \rho_i^x) (\rho_i^x - \rho_{ii}^x)} \\ \mu_{iii} = \frac{Z_{iii} - (\rho_{ii}^x + \rho_{iii}^x) Z_i + \rho_{ii}^x \rho_{iii}^x Z_0}{(\rho_{iii}^x - \rho_i^x) (\rho_{iii}^x - \rho_{ii}^x)} \end{array} \right.$$

To satisfy six observations.

This agrees with the formulas which I gave in my 20th lesson of analysis when  $n$  is set to 1. In order to make the elimination calculations which give  $\mu_i$ ,  $\mu_{ii}$ , etc., identical to those presented in the lesson, one can set  $\rho_i^x = \psi_i$ ,  $\rho_{ii}^x = \psi_{ii}$ , etc., and one can look for  $\mu_i$ ,  $\mu_{ii}$ , etc., in terms of  $\psi_i$ ,  $\psi_{ii}$ , etc. One knows that here  $n$  is the constant increase in  $x$ , or  $\Delta x$ . Once the quantities  $\rho_i$ ,  $\rho_{ii}$ ,  $\rho_{iii}$ , etc. and  $\mu_i$ ,  $\mu_{ii}$ ,  $\mu_{iii}$ , etc., are determined, their values can be substituted the equation (1).

$$Z = \mu_i \rho_i^x + \mu_{ii} \rho_{ii}^x + \mu_{iii} \rho_{iii}^x + \dots + \mu_{(n)} \rho_{(n)}^x,$$

which will then be disposed to satisfy the  $2n$  given observations, and which will be able to produce some result intermediate between those obtained in fact.

Second case, where the number of observations is odd:

In order to solve the second case, it should be noted that equation (2) differs only by the constant  $\mu_{(n+1)}$  from equation (1). Thus the series of  $\mu$  obtained from (2) is of the same nature as that obtained from (1), the only difference being that in (2) each term is increased by  $\mu_{(n+1)}$ . If these terms are therefore decreased by  $\mu_{(n+1)}$ , the remainders will satisfy the relationships which make up a series of order  $n^*$ . Thus, by preserving the notation of the previous article, one has

$$\begin{aligned}
 A_0(Z_0^{-\mu_{(n+1)}}) + A_i(Z_i^{-\mu_{(n+1)}}) + A_{ii}(Z_{ii}^{-\mu_{(n+1)}}) + \dots + A_{(n)}(Z_{(n)}^{-\mu_{(n+1)}}) &= 0 \\
 A_0(Z_i^{-\mu_{(n+1)}}) + A_i(Z_{ii}^{-\mu_{(n+1)}}) + A_{ii}(Z_{iii}^{-\mu_{(n+1)}}) + \dots + A_{(n)}(Z_{(n+1)}^{-\mu_{(n+1)}}) &= 0 \\
 A_0(Z_{ii}^{-\mu_{(n+1)}}) + A_i(Z_{iii}^{-\mu_{(n+1)}}) + A_{ii}(Z_{iv}^{-\mu_{(n+1)}}) + \dots + A_{(n)}(Z_{(n+2)}^{-\mu_{(n+1)}}) &= 0 \\
 A_0(Z_{iii}^{-\mu_{(n+1)}}) + A_i(Z_{iv}^{-\mu_{(n+1)}}) + A_{ii}(Z_v^{-\mu_{(n+1)}}) + \dots + A_{(n)}(Z_{(n+3)}^{-\mu_{(n+1)}}) &= 0 \\
 \dots & \\
 A_0(Z_{(n-1)}^{-\mu_{(n+1)}}) + A_i(Z_{(n)}^{-\mu_{(n+1)}}) + A_{ii}(Z_{(n+1)}^{-\mu_{(n+1)}}) + \dots + A_{(n)}(Z_{(2n-1)}^{-\mu_{(n+1)}}) &= 0 \\
 A_0(Z_{(n)}^{-\mu_{(n+1)}}) + A_i(Z_{(n+1)}^{-\mu_{(n+1)}}) + A_{ii}(Z_{(n+2)}^{-\mu_{(n+1)}}) + \dots + A_{(n)}(Z_{(2n)}^{-\mu_{(n+1)}}) &= 0
 \end{aligned}$$

If these equations are subtracted from one another,  $\mu_{(n+1)}$  will be eliminated, and the equations will become

---

\* The recurring series is, in fact, of order  $n+1$ , but the equation of the relationship has a root equal to unity: i.e., in the term  $\mu_{(n+1)} \rho_{(n+1)}$  one has  $\rho_{n+1} = 1$ , such that the order of the equation may be decreased by one.

$$A_0 \Delta Z_0 + A_i \Delta Z_i + A_{ii} \Delta Z_{ii} + \dots + A_{(n)} \Delta Z_{(n)} = 0$$

$$A_0 \Delta Z_i + A_i \Delta Z_{ii} + A_{ii} \Delta Z_{iii} + \dots + A_{(n)} \Delta Z_{(n+1)} = 0$$

$$A_0 \Delta Z_{ii} + A_i \Delta Z_{iii} + A_{ii} \Delta Z_{iv} + \dots + A_{(n)} \Delta Z_{(n+2)} = 0$$

$$A_0 \Delta Z_{iii} + A_i \Delta Z_{iv} + A_{ii} \Delta Z_v + \dots + A_{(n)} \Delta Z_{(n+3)} = 0$$

.....

$$A_0 \Delta Z_{(n-1)} + A_i \Delta Z_{(n)} + A_{ii} \Delta Z_{(n+1)} + \dots + A_{(n)} \Delta Z_{(2n-1)} = 0$$

For the sake of convenience, one can set  $A^{(n)} = 1$  in the numerical applications.

From those equations, the values  $\frac{A_0}{A_{(n)}}$ ,  $\frac{A_i}{A_{(n)}}$ ,  $\frac{A_{ii}}{A_{(n)}}$ , etc., can be

obtained, namely:

$$\text{For } n = 1 \dots A_0 : A_i = \Delta Z_i : \Delta Z_0$$

Value obtained from three observations.

For  $n = 2 \dots$

$$A_0 : A_{ii} = \frac{\Delta Z_i \Delta Z_{iii} - \Delta Z_{ii} \Delta Z_{ii}}{\Delta Z_0 \Delta Z_{ii} - \Delta Z_i \Delta Z_i}$$

$$A_i : A_{ii} = \frac{\Delta Z_i \Delta Z_{ii} - \Delta Z_0 \Delta Z_{iii}}{\Delta Z_0 \Delta Z_{ii} - \Delta Z_i \Delta Z_i}$$

Values obtained from five observations.

For  $n = 3 \dots$

$$A_0 : A_{iii} = \frac{+(\Delta Z_{iii} \Delta Z_{iii} - \Delta Z_{ii} \Delta Z_{iv}) \Delta Z_{iii} + (\Delta Z_i \Delta Z_{iv} - \Delta Z_{ii} \Delta Z_{iii}) \Delta Z_{iv} + (\Delta Z_{ii} \Delta Z_{ii} - \Delta Z_i \Delta Z_i) \Delta Z_v}{-(\Delta Z_{iii} \Delta Z_{iii} - \Delta Z_{ii} \Delta Z_{iv}) \Delta Z_0 - (\Delta Z_i \Delta Z_{iv} - \Delta Z_{ii} \Delta Z_{iii}) \Delta Z_i - (\Delta Z_{ii} \Delta Z_{ii} - \Delta Z_i \Delta Z_i) \Delta Z_{ii}}$$



$$A_i:A_{iii} = \frac{+(\Delta Z_i \Delta Z_{iv} - \Delta Z_{ii} \Delta Z_{iii}) \Delta Z_{iii} + (\Delta Z_{ii} \Delta Z_{ii} - \Delta Z_0 \Delta Z_{iv}) \Delta Z_{iv} + (\Delta Z_0 \Delta Z_{iii} - \Delta Z_i \Delta Z_{ii}) \Delta Z_v}{-(\Delta Z_i \Delta Z_{iv} - \Delta Z_{ii} \Delta Z_{iii}) \Delta Z_i - (\Delta Z_{ii} \Delta Z_{ii} - \Delta Z_0 \Delta Z_{iv}) \Delta Z_{ii} - (\Delta Z_0 \Delta Z_{iii} - \Delta Z_i \Delta Z_{ii}) \Delta Z_{iii}}$$

$$A_{ii}:A_{iii} = \frac{+(\Delta Z_{ii} \Delta Z_{ii} - \Delta Z_i \Delta Z_{iii}) \Delta Z_{iii} + (\Delta Z_0 \Delta Z_{iii} - \Delta Z_i \Delta Z_{ii}) \Delta Z_{iv} + (\Delta Z_i \Delta Z_i - \Delta Z_0 \Delta Z_{ii}) \Delta Z_0}{-(\Delta Z_{ii} \Delta Z_{ii} - \Delta Z_i \Delta Z_{iii}) \Delta Z_{ii} - (\Delta Z_0 \Delta Z_{iii} - \Delta Z_i \Delta Z_{ii}) \Delta Z_{iii} - (\Delta Z_i \Delta Z_i - \Delta Z_0 \Delta Z_{ii}) \Delta Z_{iv}}$$

Values obtained from seven observations.

Etc.

Etc.

Etc.

Then, if k is a positive number smaller than n, one will be able to evaluate  $\mu_{(n+1)}$  from one of the n equations which are comprised in

$$\mu_{(n+1)} = \frac{\{A_0 Z_{(k)} + A_i Z_{(k+1)} + A_{ii} Z_{(k+2)} + \dots + A_{(n)} Z_{(k-n)}\} : A_{(n)}}{(A_0 + A_i + A_{ii} + \dots + A_{(n)}) : A_{(n)}}$$

which must give the same value for  $\mu_{(n+1)}$  no matter which of the values of 0, 1, 2, 3...n one takes for k. If one then solves the equation

$$A_0 + A_i \alpha + A_{ii} \alpha^2 + A_{iii} \alpha^3 + \dots + A_{(n)} \alpha^{(n)} = 0$$

the n roots which will be obtained will be the values of  $\rho_i^{x_i}, \rho_{ii}^{x_{ii}}, \rho_{iii}^{x_{iii}}, \dots, \rho_{(n)}^{x_{(n)}}$

and the values of  $\mu_i, \mu_{ii}, \mu_{iii},$  etc., of the same equation will be calculated from the following formulas\*\*:

$$\mu_i = \frac{(Z - \rho_i^{x_i}) (Z - \rho_{iii}^{x_{iii}}) (Z - \rho_{iv}^{x_{iv}}) \dots (Z - \rho_{(n)}^{x_{(n)}}) (Z - 1)}{(\rho_i^{x_i} - \rho_i^{x_i}) (\rho_i^{x_i} - \rho_{iii}^{x_{iii}}) (\rho_i^{x_i} - \rho_{iv}^{x_{iv}}) \dots (\rho_i^{x_i} - \rho_{(n)}^{x_{(n)}}) (\rho_i^{x_i} - 1)}$$

\*\* If, in these equations, one deletes the factors  $Z-1$  and  $\rho_i^{x_i-1}, \rho_{ii}^{x_{ii}-1}, \rho_{iii}^{x_{iii}-1},$  etc., one will have the constants which are appropriate to the general term of the series  $\Delta Z_0, \Delta Z_i, \Delta Z_{ii},$  etc., i.e., one will have the constants which should multiply  $\rho_i^{x_i}, \rho_{ii}^{x_{ii}},$  etc., in the value of  $\Delta Z.$

$$\mu_{ii} = \frac{(Z - \rho_i^{x_i}) (Z - \rho_{iii}^{x_i}) (Z - \rho_{iv}^{x_i}) \dots (Z - \rho_{(n)}^{x_i}) (Z - 1)}{(\rho_i^{x_i} - \rho_i^{x_i}) (\rho_{ii}^{x_i} - \rho_{iii}^{x_i}) (\rho_i^{x_i} - \rho_{iv}^{x_i}) \dots (\rho_i^{x_i} - \rho_{(n)}^{x_i}) (\rho_i^{x_i} - 1)}$$

$$\mu_{iii} = \frac{(Z - \rho_i^{x_i}) (Z - \rho_{ii}^{x_i}) (Z - \rho_{iv}^{x_i}) (Z - \rho_{iv}^{x_i}) \dots (Z - \rho_{(n)}^{x_i}) (Z - 1)}{(\rho_{iii}^{x_i} - \rho_i^{x_i}) (\rho_{iii}^{x_i} - \rho_i^{x_i}) (\rho_{iii}^{x_i} - \rho_{iv}^{x_i}) \dots (\rho_{iii}^{x_i} - \rho_{(n)}^{x_i}) (\rho_{iii}^{x_i} - 1)}$$

.....

$$\mu_{(n)} = \frac{(Z - \rho_i^{x_i}) (Z - \rho_{ii}^{x_i}) (Z - \rho_{iii}^{x_i}) \dots (Z - \rho_{(n-1)}^{x_i}) (Z - 1)}{(\rho_{(n)}^{x_i} - \rho_i^{x_i}) (\rho_{(n)}^{x_i} - \rho_{ii}^{x_i}) (\rho_{(n)}^{x_i} - \rho_{iii}^{x_i}) \dots (\rho_{(n)}^{x_i} - \rho_{(n-1)}^{x_i}) (\rho_{(n)}^{x_i} - 1)}$$

$$\mu_{(n+1)} = Z_0 - (\mu_i + \mu_{ii} + \mu_{iii} + \dots + \mu_{(n)})$$

Observe that, as in the previous article, in the coefficients of the powers of Z in the numerators, superscripts of the same number should be substituted, or that the terms not containing Z must be multiplied by Z<sub>0</sub> and those containing Z<sub>i</sub>, Z<sub>ii</sub>, Z<sub>iii</sub>, etc., must be multiplied by Z, Z<sup>2</sup>, Z<sup>3</sup>, etc.

The last value of μ<sub>(n+1)</sub> is much easier to calculate from the previous one, which one can use only for verification. If one gives different values to n, one will have for

$$n = 1 \dots$$

$$\mu_i = \frac{Z_i - Z_0}{\rho_i^{x_i} - 1}$$

$$\mu_{ii} = Z_0 - \mu_i$$

To satisfy three observations.

n = 2...

$$\mu_i = \frac{Z_{ii} - (\rho_i^{x_i} + 1) Z_i + \rho_i^{x_i} Z_0}{(\rho_i^{x_i} - \rho_{ii}^{x_i}) (\rho_i^{x_i} - 1)}$$

$$\mu_{ii} = \frac{Z_{ii} - (\rho_i^{x_i} + 1) Z_i + \rho_i^{x_i} Z_0}{(\rho_{ii}^{x_i} - \rho_i^{x_i}) (\rho_{ii}^{x_i} - 1)}$$

$$\mu_{iii} = Z_0 - (\mu_i + \mu_{ii})$$

To satisfy five observations.

n = 3 ...

$$\mu_i = \frac{Z_{iii} - (\rho_{iii}^{x_i} + \rho_i^{x_i} + 1) Z_{ii} + (\rho_{iii}^{x_i} \rho_i^{x_i} + \rho_i^{x_i}) Z_i - \rho_{iii}^{x_i} \rho_i^{x_i} Z_0}{(\rho_i^{x_i} - \rho_{ii}^{x_i}) (\rho_i^{x_i} - \rho_{iii}^{x_i}) (\rho_i^{x_i} - 1)}$$

$$\mu_{ii} = \frac{Z_{iii} - (\rho_{iii}^{x_i} + \rho_i^{x_i} + 1) Z_{iii} + (\rho_{iii}^{x_i} \rho_i^{x_i} + \rho_{iii}^{x_i} + \rho_i^{x_i}) Z_i - \rho_{iii}^{x_i} \rho_i^{x_i} Z_0}{(\rho_i^{x_i} - \rho_i^{x_i}) (\rho_{ii}^{x_i} - \rho_{iii}^{x_i}) (\rho_{ii}^{x_i} - 1)}$$

$$\mu_{iii} = \frac{Z_{iii} - (\rho_{ii}^{x_i} + \rho_i^{x_i} + 1) Z_{ii} + (\rho_{ii}^{x_i} \rho_i^{x_i} + \rho_{iii}^{x_i} + \rho_i^{x_i}) Z_i - \rho_{iii}^{x_i} \rho_i^{x_i} Z_0}{(\rho_{iii}^{x_i} - \rho_{iii}^{x_i}) (\rho_{iii}^{x_i} - \rho_{ii}^{x_i}) (\rho_{iv}^{x_i} - 1)}$$

$$\mu^{iv} = Z_0 - (\mu_i + \mu_{ii} + \mu_{iii})$$

To satisfy seven observations.

Etc.

Etc.

Etc.

The above is still in agreement with the formulas in my 20th lesson of analysis, when in each case  $m$ , and the  $\rho$  with the highest superscript are set equal to unity.

Once the numbers  $\rho_i, \rho_{ii}, \rho_{iii},$  etc., and  $\mu_i, \mu_{ii}, \mu_{iii},$  etc. are obtained in this way, they can be inserted into equation (2)

$$Z = \mu_i \rho_i^x + \mu_{ii} \rho_{ii}^x + \mu_{iii} \rho_{iii}^x + \dots + \mu_{(n)} \rho_{(n)}^x + \mu_{(n+2)}$$

which will satisfy the  $2n+1$  given observations, and can be used to calculate all the values intermediate to those observations.

I do not consider the case where the equation

$$A_0 + A_1 \alpha + A_{ii} \alpha^2 + \dots + A_{(n)} \alpha^n = 0$$

has equal or imaginary roots; in my 20th lesson of analysis, I have given the necessary formulas for finding the solutions. It is known that equal roots introduce variable and rational coefficients in the value of  $Z$ , and if these roots are equal to unity,  $Z$  will contain entirely rational terms: thus the interpolation formulas which relate to rational functions without variable divisors, or to parabolic curves, are only a very special case of those which I have just covered.

I will now go on to applications.\*

---

\* The applications, while interesting, are not directly relevant to our purposes and have therefore not been translated.

3.0 JOURNAL ARTICLES, COMPANY AND GOVERNMENT REPORTS AND THESES

AN ANNOTATED LISTING

Auton, J.R., "Some Numerical Methods for Exponential Analysis with Connection to a General Identification Scheme for Linear Processes", M.S. Thesis, University of Kentucky, Lexington, Kentucky, 1980.

A general identification scheme for linear processes is presented and related to some well-known exponential analysis methods such as Prony's method and the pencil-of-functions method. An elucidating explanation of the source of error (bias) for the general scheme is given. Some variation of the pencil-of-functions method are given an intense examination with some revealing numerical examples. A new exponential analysis method, called the adaptive method, is developed from the general scheme. Several numerical examples of the adaptive method demonstrate its excellent performance in the presence at high levels of noise. Recommendations are made concerning the use of the adaptive method and new directions for further research.

Beatty, L.G., J.D. George, and A.Z. Robinson, "Use of the Complex Exponential Expansion as a Signal Representation for Underwater Acoustic Calibration," Journal of Acoustical Society of America, Vol. 63, No. 6, pp. 1782-1794, June 1978.

A modified Prony method is developed to provide a complex exponential signal representation for use with underwater acoustic calibration waveforms. The approach is tested using simulated waveforms, and the simulation results are validated with limited experiments. Results show the complex exponential is an ideal signal representation for underwater acoustic calibration because of its remarkable ability to extrapolate calibration waveforms beyond the actual observation period. Results are shown of real reciprocity calibration experiments that are valid down to 25 Hz where the calibration waveforms are generated from a 5-ms current ramp and where a 5-ms observation period is used. The period of the low-frequency limit is eight times the observation period used in the calibration.

Beex, A.A. (Louis) and L.L. Scharf, "Recursive Digital Filter Design Via Covariance Sequence Approximation", IEEE Transactions on Acoustics, Speech, and Signal Processing, Vol. ASSP-29, No. 1, February 1981, p. 51-57.

This paper proposes a modified solution to the general nonlinear least-squares approximation problem for digital filter design. The authors choose to approximate the covariance sequence associated with an ideal filter because it arises more naturally in spectrum matching problems than does the unit pulse response sequence. The design procedure resulting from their approach requires the rooting of an  $n^{\text{th}}$ -order polynomial, the solution of an  $n^{\text{th}}$ -order system of linear equations, and the calculation of a  $(2n-1)$  point DFT for the design of an ARMA  $(2n, 2(n-1))$  digital filter.

Berni, A.J., "Target Identification by Natural Resonance Estimation", IEEE Transactions on Aerospace and Electronic Systems, Vol. AES-11, No. 3, pp. 147-154, March 1975.

This paper presents a target identification method based on an estimation of the natural frequencies of oscillation in transient radar signatures. The emphasis is placed upon signal modeling and estimation strategy rather than relating resonance locations to physical structures. Salient features of this identification method are:  
1) target aspect angle is not needed, 2) multiple targets of the same type can be illuminated simultaneously, and 3) bandpass interrogation pulses can be used. The latter

feature is compatible with existing radar facilities. The method is applied to some simulated transfer functions, and factors affecting estimate accuracy are discussed.

Bollig, G., and K.J. Langenberg, "New Methods of Nondestructive Testing of Materials and Their Application Especially in Nuclear Engineering", Presented at the International Symposium, Paper, Saarbrücken, Federal Republic of Germany, 17-19 September 1979 (Text in German).

A new method originally developed for radar purposes is discussed for application to the acoustic scattering problem. Here the Helmholtz integral equation is Laplace transformed with respect to time for the surface pressure on a finite rotationally symmetric acoustically hard scattering body in water. The pertinent algebraic equation is obtained by Fourier series expansion with respect to the azimuthal variable and by applying the method of moments with piecewise parabolic base functions and collocation testing. The singularities of the resulting inverse matrix in the complex Laplace plane are computed for the sphere, the oblate and prolate spheroid. These poles are characteristic for the geometry of the scattering body; they can be used for classification. Applying Prony's algorithm to experimental data similar singularities are extracted.

Brittingham, J.N., E.K. Miller, and J.L. Willows, "The Derivation of Simple Poles in a Transfer Function from Real-Frequency Information", Lawrence Livermore Laboratory, UCRL-52050, 6 April 1976.

A new technique for extracting simple poles from real-frequency transfer-function data is summarized. The effects of varying pole-set parameters are illustrated with computed results, and the technique is applied to the transfer function for field scattering by a straight wire.

Brittingham, J.N., E.K. Miller, and J.L. Willows, "The Derivation of Simple Poles in a Transfer Function from Real-Frequency Information, Part 2: Results from Real EM Data", Lawrence Livermore Laboratory, UCRL-52118, 23 August 1976.

A technique for finding the simple poles in a transfer function  $F(s)$  from real-frequency ( $s=j\omega$ ) data was previously reported in the reference above and its application demonstrated for a variety of specific pole sets. This procedure, which is the analog for frequency of Prony's technique for time, is further tested against specific pole sets, and the results are described in this report.

A modification of the original procedure, which takes advantage of the fact that  $F^*(s) = F(s^*)$ , is presented. This new method is applied to electromagnetic transfer functions. The electromagnetic poles thus obtained are invariant with respect to excitation and observable phenomena (e.g., current, field, etc.) to which the transfer function corresponds, as long as the system is overdetermined. Varying the frequency interval over which data is available also demonstrates the validity of the technique.

Buckner, H.P., "Comparison of FFT and Prony Algorithms for Bearing Estimation of Narrow-Band Signals in a Realistic Ocean Environment", Journal of Acoustical Society of America, Vol. 61, No. 3, pp. 756-762, March 1977.

Samples of the quadrature field  $\{F_j\}$  collected on a uniform linear array can be quickly analyzed to determine the bearings of targets by use of the FFT or Prony algorithm. These methods are compared for many realizations of a realistic ocean acoustic signal plus background plus noise field. The familiar Monte Carlo method is used to quantify the algorithm performance. It is shown that bearing accuracy can be improved by forming the cross-sensor field  $\{G_1\}$  where  $G_1 = \langle F_j F_{j+1} \rangle$ . Theoretical and computational analysis indicate that time averaging of  $\langle F_j F_{j+1} \rangle$  not only reduces the effects of noise, but also reduces interactions of waves in the signal and background fields resulting in better array performance.



Carr, John W., "An Analytic Investigation of Transient Synthesis by Exponentials", Thesis, Massachusetts Institute of Technology, 1949.

This paper investigates least-mean-square approximations to transient functions in the time domain, offering a solution to the problem of synthesis of networks to give specified transient responses. Various exact solutions to this problem are investigated, none of which yields a solution using a high-speed electronic digital computer. Further investigation is made of useful "approximations to the best approximation." Analysis begins with the Prony method of undetermined coefficients for ordinary linear differential equations with constant coefficients, extending this method to the case of integral equations as well. This enables functions without continuous derivatives to be approximated by a set of decreasing exponentials, either real or complex.

Chan, L.C., and L. Peters, Jr., "Subsurface Electromagnetic Mine Detection and Identification", Ohio State University ElectroScience Laboratory, Technical Report 784722-1, November 1978.

The goal of this research is to adapt a commercially available radar system developed for the detection of pipes to the detection of shallow objects and further to establish a target identification capability. The target identification technique selected for this purpose (from among several being pursued at the ElectroScience Laboratory) is designated as the predictor-correlator scheme. This identification scheme has already been extremely successful in that a "minelike" target has been separated from a set of false targets. The best designed and completely tested system to date yields 100% identification of the "minelike" target and a zero false alarm rate of any of the false targets. There remains substantial improvements (some of which are nearing completion) that can be incorporated in both the radar system and in the target identification techniques. However, the target identification capability discussed in this report represents such a substantial advance that it is being published in its present form.

Chan, L.C., "Subsurface Electromagnetic Target Characterization and Identification", Ohio State University ElectroScience Laboratory, Technical Report 784722-3, April 1979.

A method for subsurface radar target characterization and identification is described. This method characterizes subsurface radar targets by their complex natural resonances which are extracted directly from their

backscattered time-domain waveforms. The difference equation coefficients associated with the complex resonances are then used in the predictor-correlator for target identification. Both the characterization and identification processes are extensively tested with real radar measurements and found to yield practical target identification performance. The target identification process is simple and involves only simple algebraic operations. Based on the identification process, a "first-generation" microcomputer identification radar system is implemented for target identification in real time. This radar system is found to yield practical identification performance.

Chan, L.C., D.L. Moffatt, and L. Peters, Jr., "A Characterization of Subsurface Radar Targets", Special Issue on Exploration Geophysics, Proc. of IEEE, July 1979.

The capability of subsurface target identification at shallow depths has been demonstrated using an electromagnetic video or baseband pulse radar. Real radar measurements were collected for five targets at a depth of 5 cm (2 in) in various ground conditions. These measurements were processed for target characterization and identification. Identification performance based on a single radar observation was evaluated. The identification process requires only simple algebraic operations and thus offers the potential of real-time on-location identification of subsurface targets. Pole patterns are extracted from the backscattered waveforms using Prony's method.

Cho, K.S. and J.T. Cordaro, "Calculations of the Singularity Expansion Method Parameters from the Transient Response of a Thin Wire", Interaction Note 379, September 1979.

The problem of determining the singularity expansion method (SEM) parameters from transient thin-wire data is examined. A computer code is used to generate response data. The SEM parameters are computed from these data using Prony's method. For noisy data, it is shown that the parameter values can be improved by signal averaging.

Chuang, C.W. and D.L. Moffatt, "Complex Natural Resonances of Radar Targets Via Prony's Method", Ohio State University ElectroScience Laboratory, Technical Report 3424-3, April 1975.

It has been shown that extraction of complex natural resonances and residues from exact transient responses via Prony's method is both efficient and accurate. However, in the art of radar interrogation the exact transient response is difficult to obtain and we must be content with an approximate one. Knowing the complex natural resonances of objects is essential to one type of radar target discrimination. In this paper we consider the use of Prony's method in extracting complex natural resonances from approximate backscattered ramp responses of radar targets.

A ramp waveform is used in radar target discrimination because a good approximation to the time domain response can be obtained with only a few harmonically related frequency domain responses via Fourier synthesis. By applying Prony's method to this approximate ramp response, a few dominant resonances which are close to the origin of the complex frequency plane can be extracted. This is quite encouraging because for the purpose of radar target discrimination only a few dominant modes are needed. Objects considered are prolate spheroids, simple wire aircraft models and some realistic aircraft models. The accuracy of this method is found to be good when compared to the results obtained by other techniques.

Chuang, C.W. and D.L. Moffatt, "Natural Resonances of Radar Targets Via Prony's Method and Target Discrimination", IEEE Transactions on Aerospace and Electronics Systems, Vol. AES-12, No. 5, pp. 583-589, September 1976.

The dominant complex natural resonances of radar targets are obtained via Prony's method applied to calculated and measured backscattered ramp response waveforms. Subject targets are spheres, simple wire models of straight and swept wing aircraft, and realistic models of modern fighter aircraft. It is demonstrated that when the backscattered ramp response waveforms are obtained via Fourier synthesis of limited spectral range harmonic scattering data, some resonance locations at variance with those obtained from reaction integral equation search procedures are obtained. It is also shown, however, that the Prony deduced resonances can be used successfully in predictor-correlator target discrimination.

Cordaro, J.T., "A Note on Representing a Transient Waveform by a Finite Sum of Complex Exponentials", AFWL-TR-77-46, Air Force Weapons Laboratory, July 1977. Also Mathematics Note 46, July 1977.

The problem of representing an observed transient waveform by a finite sum of complex exponentials is considered. A least squares technique that involves overfitting the model signal and then estimating the correct order is applied to simulated data as well as data recorded on photographs. In all the examples, the fitted waveform approximates the original waveform well when both functions are plotted on the same graph.

Cordaro, J.T., "Comparison of Three Techniques for Calculating Poles and Residues from Experimental Data," Mathematics Note 61, August 1978.

Three pole and residue calculation techniques are compared by applying the methods to simulated data. Each method works well when the data consist of a damped sinusoid plus noise. But when the data include effects from pulser asynchronism and ground reflection, accurate pole calculation is more difficult. For the methods to produce accurate results under some EMP test conditions, it is necessary to use incident field data and information about the SEM coupling coefficients and natural modes of the body exposed to EMP.

Cordaro, J.T., "Pole Measurements for the ATHAMAS Pipe Test", Mathematics Note 56, August 1977.

In the ATHAMAS pipe test, surface current and charge density response data were recorded for a metal cylinder illuminated by EMP. Poles and residues describing some of these data have been computed using an iterative technique. Some poles can be identified as corresponding to natural modes of the cylinder. Others match the incident field poles. Most poles in the incident field are shown to be due to ground reflection.

de Figueiredo, Rui J.P. and C.L. Hu, "Application of a Frequency Domain Prony Method to Wide Bandwidth Radar Signature Classification," Department of Electrical Engineering, Rice University, Houston, Texas, Technical Report EE 7908, September 1979.

Let  $x$  and  $y$  be signals (i.e., real-valued functions of time) of finite duration and energy. This paper develops a frequency domain Prony approach for interpolating, or, in general, approximating  $y(t)$  by  $\sum_{i=1}^M a_i x(t-T_i)$ ; where  $\underline{a} = (a_1, \dots, a_M)$  and  $\underline{\tau} = (\tau_1, \dots, \tau_M)$  are real  $M$ -vector parameters. If  $x$  and  $y$  denote the Fourier transforms of  $x$  and  $y$ , then  $H(\omega) = Y(\omega)/X(\omega)$  is a linear combination of exponentials with coefficients  $a_i$ ; and exponents  $-j\tau_i\omega$ , where  $j = \sqrt{-1}$ . The Prony method is used to determine  $a_i$  and  $\tau_i$  from the samples of  $H$ . These samples are obtained from the (time-domain) samples of  $x$  and  $y$  by means of the DFT. Both the cases in which these samples are noise-free and contaminated by noise are considered. Special modifications of the conventional Prony procedure to take into account the fitting of complex data are presented. These results are then extended to the case where  $y$  results from a linear combination of several known signals  $x_i$ , each differently delayed. The case in which  $y$  results from the convolution of a known signal  $x$  with an unknown impulse response  $h$  is also considered. A connection between Wiener's Tauberian theorems and the frequency domain Prony approximation are presented here.

de Figueiredo, Rui J.P. and C.L. Hu, "A Fourier-Prony Tauberian Approach to the Analysis of a Mixture of Delayed Signals," Department of Electrical Engineering, Rice University, Houston, Texas, Technical Report EE 7907, September 1979.

A frequency domain Prony approach is presented for extracting features of return signals from targets illuminated by wide bandwidth (short pulse) radar. Theoretical details pertaining to this approach are described in the above reference. The features consist of the relative delays and reflection coefficients pertaining to scattering centers on the target representing differently shaped regions on the target surface. The dimensionality of the feature vectors thus constructed is very low (less than ten). Moreover, when used in the classification of targets by a nearest neighbor classification strategy, such feature vectors permit accurate discrimination between targets that do not differ much in shape; and also they are in a large measure insensitive to noise. The results presented were corroborated by computer simulations performed on the data base created by the coherent X-band short pulse (0.5 nanosecond) radar at the Fort Worth operation Radar Range of General Dynamics Convair Aerospace Division.

Dudley, D.G., "Fitting Noisy Data with a Complex Exponential Series," Lawrence Livermore Laboratory, UCRL-52242, 7 March 1977. Also Mathematics Note 51, 7 March 1977.

The process of fitting noisy data with a complex exponential series is considered. An analysis of the classical Prony method shows that attempts to obtain accurate answers in the presence of noise meet with considerable difficulties. Recasting the problem as one of system identification and applying the least-squares method leads to biased estimates of the parameters in the characteristic equation for the system. The difficulty centers around the fact that neither classical Prony nor least squares involves any analysis, identification, smoothing, or filtering of the noise. Some techniques for improving results are discussed, and an iterative generalized least-squares procedure, which leads to a noise filter, is recommended for further study.

Dudley, D.G., "Simple Discrete System Modelling in Hilbert Space," University of Arizona, 1 February 1978.

The design of a simple discrete system model for a sequence of data is given. The model is shown to be compatible with approximation in Hilbert space. The minimum norm solution is obtained. The result is a structure which accommodates many of the results of system identification and linear prediction. The effects of error are briefly mentioned.

Dudley, D.G., "Parametric Modeling of Transient Electromagnetic Systems," Radio Science, Vol. 14, No. 3, pp. 387-396, May-June 1979.

The modeling of a transient electromagnetic system as a single-input, single-output, linear, casual process is considered. Two canonical forms are discussed: one for impulse function input and the other for more general input. Expressions are derived for errors in the estimation of the model parameters caused by errors in the process data. It is shown that Prony's method is a special case of the impulse function model and that serious errors occur in attempts to apply the method in a noisy environment. It is also shown that least squares estimation of the model parameters produces biased results and that attempts to overcome this problem have led to nonlinear equation sets. Some recommendations are made concerning noise identification and filtering in order to improve results.

Duhamel, P. and J.C. Rault, "Automatic Test Generation Techniques for Analog Circuits and Systems: A Review," IEEE Transactions on Circuits and Systems, Vol. CAS-26, No. 7, pp. 411-440, July 1979.

This paper is both a review and an assessment of techniques presently available for automatic test generation for analog systems techniques for categorizing the problems of automatic testing (definitions, faults in analog systems, different types of tests, main operations, and diagnosis procedures), characterization and description modes of analog systems, and the main software ingredients of automatic test equipment can be proposed. Several techniques, respectively, proceeding from approaches based on deterministic and probabilistic estimation, taxonomical and topological analyses are detailed. Techniques specific to linear systems (several of them belonging to the above three categories) are dealt with in a separate section. Prony's method is described as a method of transfer function determination.

Fletcher, D.L. and C.N. Weygandt, "A Digital Method of Transfer Function Calculation," IEEE Trans. Cir. Theory, Vol. CT-18, pp. 185-187, January 1971.

A method is developed that allows the direct calculation of the least-squares estimates of the poles and residues of the transfer function of a linear system when the input and output of the system is represented in a sequential numerical format. The method is particularly adaptable to implementation on a digital computer and is quite efficient for problems of moderate size. The method presented can be considered an extension to Prony's method as applied to transfer function synthesis.

Fromme, D.A., V.A.J. van Lint, R. Stettner, R.W. Macgurn, and G.M. Goldstein, "Exploding-Wire Photon Testing of Skynet Satellite," IEEE Transactions on Nuclear Science, Vol. NS-25, No. 6, pp. 1349-1357, December 1978.

This paper presents a comparison of part of the data resulting from a series of photon, electrical and calculational exercises on a structural model of the Skynet I. Calculations made for comparison with the test data were of two forms: code simulations and analytic models. The code calculations were the most detailed and included representations of the major three-dimensional geometrical features that were apparent on the Skynet Structure Model (SSM). For the most part, predictions were made at high flux, using a prescribed source and a generic pulse. Analytic models were used to understand the results produced by the code calculations and the experiments. Simple analytic circuit models representing electromagnetic modes of the system show promise of being reasonably accurate predictive means for rapid computation of the electromagnetic response.



Gardner, C.D., "Expals-Fortran Code for Exponential Approximation by Least Squares," Lawrence Radiation Laboratory, UCRL-14541, Rev. 1, University of California, Livermore, March 1970.

EXPALS is a FORTRAN program for obtaining a weighted least-squares fit of a sum of exponentials to be run on CDC 6600 or IBM 7094 computers. This routine fits a linear combination of real exponential decay functions by the Prony-Householder iteration method. This report presents the mathematics and programming necessary to run EXPALS. Also included are output of examples to test the program.

Gavel, D.T., J.V. Candy, and D.L. Lager, "Parameter Estimation from Noisy Transient Electromagnetic Measurement," Lawrence Livermore Laboratory, Submitted for Nuclear EMP Meeting, Anaheim, California, 5-7 August 1980.

This paper describes the performance of three different signal processing techniques applied to parameterize a body from noise experimental electromagnetic transient response data. The techniques range from the well-known Prony method to the more sophisticated extended Kalman filter and finally to the highly sophisticated maximum likelihood identifier. Comparison is made of the performance of these algorithms and a discussion of their tradeoffs is given.

Gerck, E., "Prony Method Revised," Applied Optics, Vol. 18, No. 18, pp. 3075, 15 September 1979.

The Prony method described in its application to optical resonators by Siegman and Miller has since then deserved some attention. This letter shows that the Prony method for symmetric matrix is all that is necessary for any scalar resonator calculation, regardless of misalignments or misfiguring of mirrors. This is accomplished by simple modifications of the original algorithm. In this way a simpler and faster procedure is presented only by considering the correct scalar product to be used. The proposed method, however, could not possibly apply directly in the cases for which the polarization of the beam is changed as in "optical diodes" or axicons.

Goodman, D.K., "Determination of the Number of Terms Necessary for a Class of Approximation Procedures," in Data Representation, Edited by H.S. Andersen and M.R. Osborne, University of Queensland Press, pp.77-93, 1970.

This presentation is concerned with the following areas: (1) the order determination task when the basis functions are exponentials; (2) a tutorial introduction to a new arithmetic designed for digital computation which produces automatic error bounding; and (3) a solution, both theoretical and practical, to the problem in (1) and the exponential fitting procedure.

Henderson, T.L., "Matrix Methods for Determining System Poles from Transient Response", University of Kentucky, Electromagnetics Research Report 80-1, May 1980. Also Mathematics Note 66, May 1980.

The problem treated is that of identifying the poles of a finite order system by observing its transient decay after cessation of input, for a limited time, using (possibly) multiple observation points and experimental repetition. Various approaches are studied, having the common characteristic that a homogeneous matrix equation must be solved. Several techniques that have been given scant attention in the literature are consolidated into the treatment, together with new results including an analytical treatment of the consequences of assuming an excessively high system order, derivation of a statistically unbiased estimate for an intermediate parameter in the solution, new theorems on error effects, a recipe for effective use of the singular value decomposition, a new method for suppression of extraneous poles, an elucidating derivation and extension of the method of Jain, a new form of the problem wherein the system poles are eigenvalues, and a study of the relationship between various pole identification methods.

Henderson, T.L., "Geometric Methods for Determining System Poles from Transient Response", to be published by IEEE Transaction on ASSP.

The problem treated is that of identifying the poles of a finite order system by observing its transient decay after cessation of input, for a limited time, using (possibly) multiple observation points and experimental repetition. Various approaches are studied, having the common characteristic that a homogenous matrix equation must be solved. Several techniques that have been given scant attention in the literature are consolidated by a geometric treatment, together with new results including an analytical treatment of the consequences of assuming an excessively high system order, derivation of a statistically unbiased estimate for an intermediate parameter in the solution, new theorems on error effects, a recipe for effective use of the singular value decomposition, and a new method for suppression of extraneous poles.

Hiller, J., "Exponential Fitting and General Techniques of the Prony Form", in Data Representation, edited by Andersen and Osborne, University of Queensland Press, pp. 62-76, 1970.

The intention of the first part of this paper is to pose a series of queries and to present some examples that serve to show the relevance of exponential fitting. It is demonstrated that the justification for exponential

fitting is often poor. Nonetheless it is not always easy to pose the "correct" fitting problem and exponential fitting may be reasonably invoked by default. The second part of the paper discusses the Prony procedure in terms of commuting operators. The aim of the treatment here is to point out the questions that should be asked so that general operator classes may be handled.

Holt, John N. and Robyn J. Antill, "Determining the Number of Terms in a Prony Algorithm Exponential Fit," Mathematical Biosciences, 36, pp. 319-332, 1977.

The problem of determining the correct number of terms to take in an exponential sum approximation to equally spaced data is considered. The Prony algorithm is used, together with singular value decomposition techniques. A criterion is established which is useful in estimating the number of terms when the data are subject to normally distributed, zero mean noise.

Householder, A.S., "On Prony's Method of Fitting Exponential Decay Curves and Multiple-Hit Survival Curves," Oak Ridge National Laboratory, ORNL-455, Oak Ridge, Tennessee, February 1950.

This paper first describes Prony's method. Thereafter it is shown how one can proceed to obtain a valid least squares fit, and finally a criterion for choosing the number of exponentials required is given. This is the source of the Householder orthogonalization procedure.

Hudson, H.G. and D.L. Lager, "Observations on the Operation of the SEMPEX Code," Lawrence Livermore Laboratory, UCID-17440, 22 September 1976.

The authors' observations on the use of the SEMPEX computer code are given. The issues of proper sampling rate and total sampling interval are addressed. The effect that noise in the data has on the results and the ability of autocorrelation, truncation filtering, and the sliding-window technique to reduce these effects are also discussed.

Jain, V.K., "Representation of Sequences," IEEE Transactions on Audio and Electroacoustics, Vol. AU-19, No. 3, pp. 208-215, September 1971.

In this paper, the space of square summable real sequences is considered. Insight is developed into the structure of the space and a method for representation of its elements is described. The method is shown suitable for on-line implementation and is noniterative; it can be used for waveform design, signal feature extraction, and discrete-time system identification. The applications are exemplified by three simulation studies.

Jain, Vijay K., Tapan K. Sarkar, and Donald D. Weiner, "Extension of Pencil-of-Functions Method to Reverse-Time Processing with First-Order Digital Filters," Department of Electrical Engineering, Rochester Institute of Technology, Rochester, New York, Technical Report No. 3, August 1980.

In this presentation, the data signal is processed in reverse-time by a cascade of first order digital filters to yield a family of information signals. The Gram matrix of these information signals is shown to contain the essential information on the poles of the signal. The entire procedure of the application of pencil-of-function method is thus noniterative. Examples presented demonstrate (i) noiseworthiness in the representation problem when data are corrupted by noise, and (ii) the effectiveness of the method in the approximation problem.

Kammler, D.W., "Existence of Best Approximations by Sums of Exponentials," Journal of Approximation Theory, Vol. 9, No. 1, pp. 78-90, September 1973.

This thesis studies the problem of approximation by sums of exponentials and contains a discussion of Prony's method.

Kammler, D.W., "Prony's Method for Completely Monotonic Functions," Journal of Mathematical Analysis and Applications, Vol. 57, No. 3, pp. 560-570, 1 March 1977.

Prony's method for constructing an approximating exponential sum requires one to select  $2n$  uniformly spaced points, solve a difference equation to find  $c_1, \dots, c_n$ , find the roots of a polynomial and take a log to find the exponents  $\lambda_1, \dots, \lambda_n$ . This procedure cannot be used to produce an exponential sum for an arbitrary  $F \in C[0, \infty]$  but then one cannot reasonably expect to model an arbitrary continuous function with such an exponential sum. This paper proceeds under the additional hypothesis that  $F$  is completely monotonic on  $[0, \infty]$ . It is shown that Prony's method always yields an exponential sum which for sufficiently large  $n$  can be made arbitrarily close to  $F$  (in the uniform norm) if and only if  $F$  is completely monotonic on  $[0, \infty]$ .

Kelso, J.M., "Measuring the Vertical Angles of Arrival of HF Skywave Signals with Multiple Modes," Radio Science, Vol. 7, No. 2, pp. 245-250, February 1972.

The measurement of the vertical angle of arrival of HF skywave, multimode signals generally requires either highly complicated mathematical manipulations or observing techniques that necessitate a waiting period until the relative phases among the modes either reach a desired condition or pass through a desired range of variations. This paper utilizes an extension of a very old method (the Prony method, c. 1795) for approximating sums of exponential functions. The technique uses relatively simple calculations that are based on an instantaneous (or nearly so) set of samples from a uniformly spaced, vertically disposed array of antenna elements. Unlike many other methods, this technique requires neither time variations of phase nor a perfectly reflecting ground plane, and is valid for arbitrary polarization of the (similar) antenna elements and for any general wave polarization. Experimental results from some field tests of the process for vertical angle estimation are presented.

Khatwani, K.J., and J.S. Bajwa, "Identification of Linear Time-Invariant Systems Using Exponential Signals," IEEE Transactions on Automatic Control, pp. 146-148, February 1975.

A new method of system identification for linear time invariant systems based on Prony's method of exponential interpolation is suggested in this correspondence. This method of identification uses an exponential signal for system excitation and has the advantages of simplicity, independent determination of numerator and denominator of the system transfer without any difficulty. The simplicity of the method has resulted because of the simple but particularly suitable formulation in the state variable form used.

Kremer, H. and A. Lacroix, "Praktische Berechnung von Exponentialapproximationen mit dem Prony-Verfahren", Angewandte Informatik, Vol. 3, pp. 121-126, 1972.

Prony's method is described for the approximation of discrete sets of points by exponential functions with complex argument. The determination of the degree of the approximating function is discussed in detail. It is shown by examples how input errors influence the results of the method. The algorithm has been formulated in ALGOL and tested on an IBM 7040 computer; the ALGOL program is given.

Kremer, H., "Design of Linear Networks with Prescribed Impulse Response", Extended paper from IEEE Conference on Computer-Aided Design of Electronic and Microwave Circuits and Systems, Hull, United Kingdom, 12-14 July 1977.

A method for the design of linear networks with prescribed impulse response is presented, that uses Prony's algorithm for the approximation and gives the network as a parallel structure of at most second order subnetworks. Following a brief description of the algorithm some numerical and computational aspects are discussed. The method is illustrated by two examples, the design of a Hilbert filter and an ideal low-pass filter.

Kulp, R.C., "An Optimum Sampling Procedure for Use with the Prony Method", IEEE Trans. on Electromagnetic Compatibility, Vol. EMC-23, No. 2, May 1981, pp. 67-71.

The ill-posed nature of identifying the poles of a transient response is discussed along with a sampling scheme for minimizing the effect of this ill-posedness. Two simulation examples are given. The results of the second example are compared to previously published results.

Kunz, K.S., et al., "Surface Current Injection Techniques: A Theoretical Investigation", IEEE Transactions on Nuclear Science, Vol. NS-25, No. 6, pp. 1422-1427, December 1978.

This theoretical study was undertaken to find surface current injection technique configurations capable of quality simulation. A configuration capable of quality simulation was found. It compared favorably with conventional EMP simulation techniques when both were measured against theoretical threat aircraft responses. Some preliminary experimental results further confirm the capabilities of this configuration while validating the theoretical predictions. Prony's method was used to find the poles and residues of the data.

Kunz, K.S. and J.F. Prewitt, "Practical Limitations to a Natural Mode Characterization of Electromagnetic Transient Response Measurements," IEEE Transactions on Antennas and Propagation, Vol. AP-28, No. 4, pp. 575-577, July 1980.

Practical limitations arising from limited signal-to-noise ratios in actual transient electromagnetic measurements limits the number of natural modes which can be extracted from such measurements. This number is shown here to be three to six for a complex scatterer, in this case an aircraft, five for a "fat" cylinder and thin wire. This is for double exponential waveform excitation of the aircraft and step function excitation of the fat cylinder and thin wire. A few more modes can be resolved if the excitation is altered so as to provide more high frequency energy, as can be accomplished by changing the step excitation to a delta function excitation, or by signal conditioning, for example making a derivative measurement that emphasizes the high frequency response. Even so, far fewer modes can be resolved experimentally than can be predicted. Experimental electromagnetic transient response measurements should therefore be made with these practical limitations in mind.

Lager, D.L., H.G. Hudson, and A.J. Poggio, "User's Manual for SEMPEX: A Computer Code for Extracting Complex Exponentials from a Time Waveform", Mathematics Note 45, March 1977.

This report is the user's manual for the SEMPEX computer code. Singularity Expansion Method Pole Extraction is a technique in electromagnetics in which the free response of a structure is expressed as a weighted sum of complex exponentials where the damping factors of the exponentials are derived from the poles (i.e., singularities) of the transfer function of the object.

Lager, D.L., "Using Paret (Parameter Estimation Technique) for the Determination of Mode Shapes," Lawrence Livermore Laboratory, UCID-17898, 8 September 1978.

This report applies the Parameter Estimation Technique (PARET) to the measured responses of a T-shaped aluminum plate. The frequency and damping for the natural resonances and display animated mode shapes for the plate are obtained. The measurements consist of the response of an accelerometer mounted on a corner of the plate to hammer blows at 38 separate locations on the plate. PARET is used to compute the natural resonances by determining the parameters of an exponential model that fits the responses. Tabulation of the frequencies and damping for seven modes occurring over a frequency range 500-1650 Hz and also plot the shape for each mode are made.

Lager, D.L., "A Statistical Evaluation of the Performance of the Time-Domain Paret Algorithm," Lawrence Livermore Laboratory, UCID-18120, 1 June 1979.

A methodology was developed for evaluating the performance of the time-domain PARET (TDP) algorithm when processing waveforms contaminated with additive noise. The methodology is general; it may be applied to evaluating the performance of other algorithms and provides a good basis for comparison. The performance is evaluated by computing the bias, variability, and root mean square error in the parameters estimated by the algorithm when processing ensembles of waveforms. The waveforms used in this study were the displacements as a function of time from a computer simulation (by the SAP IV Program) of an eight-floor structure.

Lai, Hang H. and John D. Dyson, "The Determination of the Direction of Arrival of an Interference Field", Radio Science, Vol. 16, No. 3, pp. 365-376, May-June 1981.



The problem of resolving the complex interference field at a high receiving site, that is caused by multimode propagation at high frequencies, into the individual plane wave components and determining the direction of arrival of these components is considered. The method developed is based on Prony's algorithm. To illustrate the effect of noise on the solution, a first-order linearized model of a statistical study of a two plane wave problem for this conventional Prony's method is presented. A modification of Prony's technique, an eigenvalue approach, allows one to analytically determine the number of waves in the problem, and this approach is less susceptible to the effect of noise. Data obtained from a controlled experiment when a signal from a distant source was repeated and retransmitted at a different location verify that the system as well as the algorithm is capable of resolving waves coming from different directions. Experimental evidence related to multimode as well as single-mode propagation coming from the same source has been obtained.

Leveckis, A.S., "Application of Singularity Expansion to Electromagnetic Pulse Response Waveforms," Master's Thesis, Massachusetts Institute of Technology, June 1977.

Here is presented the results of an attempt to use Prony's method on EMP response waveforms on ships. The technique to acquire the experimental data is detailed; the numerical problem is posed, and the goals are quickly simplified to attempting to find the best curve-fit to the waveform, using complex exponentials, and then hoping to eliminate the curve-fit poles. A method is presented which aids in the 'physical pole' selection process, and explained also is the significance of the right half plane poles in the results.

Lytle, R.J. and D.L. Lager, "Using the Natural-Frequency Concept in Remote Probing of the Earth," Radio Science, Vol. 11, No. 3, pp. 199-209, March 1976.

A technique is given for determining the thicknesses and the electrical constitutive parameters of a planar-layered medium such as a coal seam in a mine environment. Time-domain experimental data are analyzed with Prony's method to determine the natural frequencies of the layered medium. Explicit relations are given (for dielectric layers) for determining the thicknesses and dielectric constants from the experimentally determined natural frequency results. Explicit expressions are also given (for conductive layers) for calculating the electrical thickness from natural frequency results. The method is illustrated with sample numerical results.

Makhoul, J., "Linear Prediction: A Tutorial Review", Proceedings of the IEEE, Vol. 63, No. 4, pp. 561-580, April 1975.

This paper gives an exposition of linear prediction in the analysis of discrete signals. The signal is modeled as a linear combination of its past values and present and past values of a hypothetical input to a system whose output is the given signal. In the frequency domain, this is equivalent to modeling the signal spectrum by a pole-zero spectrum. The major part of the paper is devoted to all-pole models. The model parameters are obtained by a least squares analysis in the time domain. Two methods result, depending on whether the signal is assumed to be stationary or nonstationary. The same results are then derived in the frequency domain. The resulting spectral matching formulation allows for the modeling of selected portions of a spectrum, for arbitrary spectral shaping in the frequency domain, and for the modeling of continuous as well as discrete spectra. This also leads to a discussion of the advantages and disadvantages of the least squares error criterion. A spectral interpretation is given to the normalized minimum prediction error. Applications of the normalized error are given, including the determination of an "optimal" number of poles. The use of linear prediction in data compression is reviewed. For purposes of transmission, particular attention is given to the quantization and encoding of the reflection (or partial correlation) coefficients. Finally, a brief introduction to pole-zero modeling is given.

Maria, G.A. and M.M. Fahmy, " $l_p^m$  Approximation by Exponentials", IEEE Transactions on Circuit Theory, pp. 71-74, January 1973.

An iterative method is proposed for obtaining a solution to the problem of  $l_p^m$  approximation by exponentials. The method always converges and has the advantage that the samples need not be equidistant. Using the proposed method and allowing  $p$  to increase yields, in a much easier way, results that are comparable to those obtained by Fischl for the Chebyshev approximation.

Marzollo, A., "On the Mean Square Approximation of a Function with a Linear Combination of Exponentials", Int. Journal Control, Vol. 9, No. 1, pp. 17-26, 1969.

This paper deals with the problem of finding the  $2N$  coefficients of a linear combination of  $N$  exponentials which minimizes its mean square error from a given function  $f(t)$ . The real difficulty lies in the determination of the optimal exponents.

A real function of  $N$  real variables is constructed, such that the points where it is minimal give the optimal exponents. Such a function may be considered as the sample for  $t \in T$  of the response to  $f(T, t)$  of a simple system. This implementation allows a significant simplification of the given problem from a computational point of view, for instance by hybrid methods.

Miller, E.K., et al., "Radar Target Classification Using Temporal-Mode Analysis", Lawrence Livermore Laboratory, UCRL-51825, 27 May 1975.

Exploratory test results are presented for a new radar target-classification technique based on processing the transient scattered waveform from a target using temporal-mode analysis. The new method offers the potential advantages of target characterization independent of aspect angle and polarization, a means for quickly comparing targets against a stored library, and substantially simplifying data storage and processing requirements. Although several theoretical questions remain to be answered, initial test results look quite promising for this new approach.

Miller, E.K., J.N. Brittingham, and J.L. Willows, "The Derivation of Simple Poles in a Transfer Function from Real Frequency Information Part 3: Object Classification and Identification", Lawrence Livermore Laboratory, UCRL-52211, 10 January 1977.

This report is the third in a series of three that evaluate a technique (frequency-domain Prony) for obtaining the poles of a transfer function. The main objective was to assess the feasibility of classifying or identifying ship-like targets by using pole sets derived from frequency-domain data. A predictor-correlator procedure for using spectral data and library pole sets for this purpose was developed. Also studied was an iterative method for reducing the effects of noise and a technique based upon magnitude-only spectral data.

Miller, E.K., "Data Characterization and Compression", Lawrence Livermore Laboratory, UCID-17511, 6 July 1977.

The information-transformation process based upon Prony's method has been found to be increasingly useful for application to electromagnetic-type problems in particular and a growing variety of physical problems in general.

Miller, E.K. and D.L. Lager, "Information Extraction Using Prony's Method", Lawrence Livermore Laboratory, UCRL-52329, 23 August 1977.

This report considers the questions of information content and data transformation. To define information content, the concepts of data rank (number of poles in the exponential data) and data precision are used. To study data transformation, a waveform is defined in terms of specified poles and residues, is sampled to obtain the associated data, and is operated on using Prony's method to transform the data and compare it with the original information content. Other numerical processes, such as matrix inversion, can be viewed in the same way. The problems of ill-conditioning are related more to the data than to the process. Finally, applications of Prony's method to other physical problems with exponential solutions are considered.

Miller, E.K. and D.L. Lager, "Prony's Method for the Angle Domain", Lawrence Livermore Laboratory, UCID-17502, Rev. 1, 6 October 1977.

Prony's method, a technique for obtaining the parameters of a series of complex exponentials, is used here to analyze the radiation pattern of a linear array of isotropic radiators. The locations and amplitudes of the radiators are derivable from the pattern. Possible applications include imaging and array synthesis.

Miller, E.K. and D.L. Lager, "Radiation Field Analysis and Synthesis Using Prony's Method", Lawrence Livermore Laboratory, UCRL-80532, 16 January 1978.

Prony's method, a technique for extracting the parameters of an exponential series from its discrete samples, is applied here to far-field pattern data. The method offers possibilities in imaging source distributions and in array analysis and synthesis.

Miller, E.K., "Prony's Method Revisited", Lawrence Livermore Laboratory, UCRL-52590, 18 October 1978.

This report continues evaluation of the moving-window approach for reducing the deleterious effects of noise to which Prony processing is particularly sensitive. A series of systematic computer experiments have been conducted on analytically specified transient waveforms and, in particular, have examined transmission of errors through the process by determining the accuracy of the input data, the characteristic-equation coefficients and roots, and the poles. Statistical distributions of these quantities for combinations of input data parameters have been derived.

Miller, E.K. and D.L. Lager, "Inversion of One-Dimensional Scattering Data Using Prony's Method", Lawrence Livermore Laboratory, UCRL-52667, 12 February 1979.

A one-dimensional configuration is the simplest geometry to invert, yet it has practical application to such problems as scattering from inhomogeneous half-spaces and propagation on non-uniform transmission lines. Whether the medium parameters vary continuously or discretely with position, the problem's numerical description can usually be developed in finite-time approximation. As such, the scattered and transmitted fields can be represented as exponential series, whose exponents are related to the electrical thicknesses of the layers which make up the model. If the exponents or poles are derivable from field data, then the inverse problem is formally solvable. This report considers application of Prony's method, a procedure for obtaining the poles of exponential signals, to such one-dimensional problems. Analysis of both time-domain and frequency-domain data is studied. The effects of the medium characteristics, number of layers, and other factors are examined. It is concluded that Prony's method has merit for certain classes of one-dimensional inverse problems.

Miller, E.K., "A Study of Target Identification Using Electromagnetic Poles", Lawrence Livermore Laboratory, UCRL-52685, 15 March 1979.

Three techniques to identify radar targets by using their electromagnetic resonances or poles are studied. A singular advantage of such a pole-based approach is that it can operate independently of the target excitation. A library of pole sets is used both to generate the target waveforms (to which noise is added) and to correlate with parameters extracted from those waveforms. The techniques studied are linear prediction, residue calculation, and pole calculation. Direct waveform correlation was also evaluated for comparison with the pole-based techniques. Using 10-run-average correlation values, error-free identification was accomplished by pole calculation at a peak-signal-to-peak-noise ratio as low as 10, and by waveform correlation at a noise ratio of unity.

Miller, E.K., D.L. Lager and J.T. Okada, "Imaging of Linear Source Distributions", Lawrence Livermore Laboratory, UCRL-52822, November 1979.

An approach based on Prony's method is used to find the spatial poles of straight wires of varying length and radius from the far-field patterns. For the antenna case, poles having the largest residues are found near the feed

region and antenna ends; smaller ones are found outside these regions. The poles are attributed to radiation from the exciting source and wire ends, due to charge acceleration, and from a slightly damped traveling wave. In the scattering case, the poles with the largest residues are located at the wire's ends for near-axial incidence and are again due to end radiation. By examining the residue magnitudes as a function of wire radius, a correlation is found with radiation from the feed region of an infinite antenna and from the ends of a semi-infinite antenna.

Miller, G., "Representation and Analysis of Signals; Part XXVI, Least-Squares Approximation of Functions by Exponentials", Department of Electrical Engineering, The John Hopkins University, Baltimore, Maryland, June 1969.

The approximation of an analytic time function in the least-squares sense by sums of exponentials is considered from several different points of view. McDonough's method for finding the poles is derived by three different approaches. In the process, a new method is developed which offers the advantages of the earlier results achieved by McDonough and by McBride, Schaeffgen, and Steiglitz. This new method reveals the link between these earlier methods and provides a standard for comparing these two linear iterative schemes using several numerical examples.

Mitra, R. and C.O. Stearns, "Field-Strength Measurements in a Multipath Field Using Linear and Circular Probing", Radio Science, Vol. 2, No. 1, pp. 101-110, January 1967.

Techniques for resolving the components of a multipath field are investigated in this paper. Both linear and circular probing schemes are discussed. A method for analyzing the linearly probed data is given, which is considerably simpler than the procedure followed by Watterson in an earlier paper. The method is also extended to the circular case. It is found that the resolution in the circular case is poorer than for the linear scheme.

Moffatt, D.L. and K.A. Shubert, "Natural Resonances Via Rational Approximations", IEEE Transactions on Antennas and Propagation, Vol. AP-25, No. 5, pp. 657-660, September 1977.

An approach for locating the dominant complex natural resonances (poles) of scatterers or radiators is described. Admittance values, calibrated using Richmond's thin wire computer programs, at a regular grid of complex and real frequencies are fit by a rational function. Zeros of the resultant numerator and denominator polynomials then yield,

respectively, the zeros and poles of the structure. The method is illustrated for resonances of straight and bent wire structures.

Moffatt, D.L., L.C. Chan and G.A. Huwisher, "Characterization of Subsurface Electromagnetic Soundings", Ohio State ElectroScience Laboratory, Annual Technical Report 4490-1, September 1977.

In this report, details on a first year's efforts to characterize subsurface radar target signatures are summarized. The report covers three main topics; an initial measurement program to obtain transient radar signatures of buried objects, a description of a comprehensive computer program to extract, by various means, the complex natural resonances of buried objects, and the complications involved in extending a synthetic radar imagery technique to buried targets. In a conclusions section, the state of the art is summarized and research for the next year's effort is discussed.

Moffatt, D.L. and L.C. Chan, "Characterization of Subsurface Soundings", Ohio State ElectroScience Laboratory, Final Report 4490-2, January 1979.

Research on a National Science Foundation Grant for the period 1 June 1976 to 31 December 1978 is summarized. Unique new excitation invariant parameters (complex natural resonances) of subsurface anomalies are described. Methods for extracting and exploiting these excitation invariant parameters in the presence of noise and clutter are given. State-of-the-art subsurface radar systems are illustrated. Using such a system, target characterization procedures are successfully illustrated for a variety of subsurface targets using real radar data.

Moffatt, D.L. et al, "Transient Response Characteristics in Identification and Imaging", IEEE Transactions on Antennas and Propagation, Vol. AP-29, No. 2, pp. 192-205, March 1981.

Three different solutions to problems in identification and inverse scattering using restricted far-zone scattering data are reviewed and illustrated. The data are restricted in the sense that the frequencies, aspects, and polarizations are limited. All three solutions are based in part on a fundamental time domain viewpoint whereby the far-zone scattering characteristics of any finite object are uniquely summarized by its impulse response waveforms. Insight provided by these waveforms is exploited via geometrical characteristics extracted from steady-state, forced, and free responses of the object. For the present purposes the fundamental importance of signaling waveforms whose

wavelengths are within an order of magnitude of the object dimension is demonstrated. The basic methods discussed have been previously identified in the literature as natural resonance estimation, radar imaging from ramp response signatures, and low frequency classification. Related common features, additional insight, and some new results are given. Applications of all three methods to objects ranging from simple to complex geometries are described.

Murphy, W.D. and M.L. Bernabe, "Numerical Procedures for Solving Nonsymmetric Eigenvalue Problems Associated with Optical Resonators", Applied Optics, Vol. 17, No. 15, pp. 2358-2365, 1 August 1978.

The Prony method is extended to handle the nonsymmetric algebraic eigenvalue problem and improved to search automatically for the number of dominant eigenvalues. A single iterative algorithm is given to compute the associated eigenvectors. Resolution studies using the QR method are made in order to determine the accuracy of the matrix approximation. Numerical results are given for both simple well-defined resonators and more complex advanced designs containing multiple propagation geometries and misaligned mirrors.



McDonough, R.N., "Representation and Analysis of Signals, Part XV: Matched Exponents for the Representation of Signals," The Johns Hopkins University, Department of Electrical Engineering, Baltimore Maryland, 30 April 1963.

Approximation by sums of exponentials is considered from various points of view, such as the time-domain approximation problem of network synthesis, in which an analytic time function is to be approximated, or situations in which discrete measured values of some dynamic process are to be smoothed and fitted using an exponential model. Prony's method for approximate determination of the exponents is exhaustively reviewed in all its forms.

McDonough, R.N. and W.H. Huggins, "Best Least-Squares Representation of Signals by Exponentials," IEEE Transactions on Automatic Control, Vol. AC-13, No. 4, pp. 408-412, August 1968.

In this paper the approximation of a given real time function over  $(0, \infty)$  by a linear combination of a given number  $n$  of exponentials is considered, such that the integrated squared error is minimized over both the  $n$  coefficients of the linear combination and the  $n$  exponents used. The usual necessary condition for stationarity of the integrated squared error leads to a set of  $2n$  simultaneous equations, nonlinear in the exponents. This condition is interpreted in the geometric language of abstract vector spaces, and an equivalent condition involving only the exponents, with the coefficients suppressed, is developed. It is next indicated how this latter condition can be applied to signals which are not known analytically, but only, for example, as voltages recorded on magnetic tape, or as a table of sampled values. The condition still in effect requires solution of nonlinear algebraic equations, and a linear iterative method is proposed for this purpose. Finally, the procedure is illustrated with a simple example.

Osborne, M.R., "A Class of Nonlinear Regression Problems", in Data Representation, University of Queensland Press, pp. 94-100, 1970.

This paper considers the problem of determining the parameters  $\alpha$ ,  $\beta$  in the model equation

$$y(t) = \sum_{i=1}^k \alpha_i f_i(t, \beta_1, \dots, \beta_m)$$

from observation

$$y_i = y(t_k) - \epsilon_i$$

made at a sequence of distinct time points  $t_1, t_2, \dots, t_n$ . It is assumed that the  $\epsilon_i$  (the experimental errors) are  $n$  independent, normally distributed, random variables with mean zero and standard deviation  $\sigma$  so that least squares methods are appropriate. It is also convenient to assume that the functions  $f_i(t, \beta)$  form a Chebyshev set for the values of  $\beta$  of interest. This means that no nontrivial linear combination of the  $f_i$  can vanish identically on any set of  $k$  distinct values of  $t$ . It is convenient to refer to a regression problem of the above form as separable, and refer to  $\alpha$  and  $\beta$  as the linear and nonlinear parameters respectively.

Parsons, D.H., "Biological Problems Involving Sums of Exponential Functions of Time: A Mathematical Analysis that Reduces Experimental Time", Mathematical Biosciences, #2, pp. 123-128, 1968.

The analysis given in this article, which is based on the theory of difference equations, enables one to determine the integer rank in advance, and then determine the  $2p$  unknown constants by means of simultaneous linear equations.

Paul, A.K., "Anharmonic Frequency Analysis", Mathematics of Computation, Vol. 26, No. 118, pp. 437-447, April 1972.

A new numerical method of frequency analysis is described, designed mainly to search for discrete frequencies in a time series. An integral transform is applied twice to the data for different reference times. A complex amplitude within a selected narrow frequency band is obtained for each transform. The frequency is then determined from the phase change of the complex amplitude over the different of the two reference times. Very high precision is obtained, which is demonstrated in two examples.

Pearson, L.W., M.L. Van Blaricum, and R. Mittra, "A New Method for Radar Target Recognition Based on the Singularity Expansion for the Target", Proceedings of the 1975 IEEE International Radar Conference, Arlington, Virginia, pp. 452-456, 21-23 April 1975.

The results of an initial study of the use of the recently introduced Singularity Expansion Method (SEM) characterization of a scatterer as a means of radar target identification are reported. Three significant advantages of this method over target identification schemes are explored. These are 1) the aspect-dependence of transient backscattered waveforms can be suppressed, 2) the transmitted waveform requirements are only moderately demanding on hardware, and 3) the SEM poles constitute a smaller collection of numbers to be presented to pattern recognition algorithms. Examples are presented indicating the following features of the method: viability in the presence of noise; excitability of a reasonable number of natural frequencies with a moderate bandwidth pulse; and the ability of a few poles to identify a target from among a collection of known targets.

Pearson, L.W. and D. R. Roberson, "The Extraction of the Singularity Expansion Description of a Scatterer from Sampled Transient Surface Current Response", IEEE Transactions on Antennas and Propagation, Vol. AP-28, No. 2, pp. 182-190, March 1980.

A method is presented whereby one can extract the singularity expansion method (SEM) description of an object's electromagnetic scattering response from spatially sampled transient surface currents. The currents are excited by a known excitation. The SEM data are recoverable to the degree that spatial coupling and (frequency) spectral intensity excite a given SEM mode. Results of a numerical study of the method using the transient response of a thin wire are reported. The data used in the study were computed using a time domain integral equation technique. The ultimate utility of the method lies in the recovery of SEM data from measured data, thus admitting complex-shaped objects into the realm of SEM description. The method is based on a Prony-type pole/residue extraction procedure.

Pearson, L.W. and D.R. Roberson, "Extrapolation of Induced Transient Surface Currents on a Scatterer to General Band-Limited Excitation Conditions", University of Kentucky Electromagnetics Research Report, pp. 1-49, October 1978, also Interaction Note 360, October 1978.

A technique for the extraction of an approximate singularity expansion description of the electromagnetic response of a scatterer is presented. The description is extracted from transient waveforms which are spatial samples of the transient current on the scatterer excited by a known excitation. The method can use any reliable Prony-type processor for exponential analysis of the transient waveforms as long as it provides physically-meaningful poles from the waveforms. The fact that singularity expansion poles are common to all waveforms on the object provides a computational advantage over a waveform-by-waveform Prony-type analysis. A "consensus pole set" procedure is described to exploit this redundancy. Some issues relating to the completeness of a set of transient data to the class of excitations to which the singularity expansion is applied is discussed. The one-dimensional example of a thin-wire scatterer is used to test the practicability of the method. The ultimate utility of the method is demonstrated by way of expansion of the extracted data to new excitations. Recommendations are made toward an implementation of the method using measured data.

Perkins, J.F. and Charles Cason, "Effects of Small Misalignments in Empty Unstable Resonators", Applied Physics Letters, Vol. 31, No. 3, pp. 198-200, 1 August 1977.

Effects of misalignment of rectangular-cross-section unstable resonators are investigated by calculations of eigenvalues and shapes of transverse modes of empty strip resonators. Small misalignments can substantially affect

mode-loss separation. A design criterion is suggested for choice of Fresnel numbers such that effects of misalignments are minimized.

Perl, W., "A Method for Curve-Fitting by Exponential Functions", International Journal of Applied Radiation and Isotopes, Vol. 8, pp. 221-222.

A method is presented for fitting exponential functions to data of the decay type. In an elementary "peeling-off" form of the method, the local ordinate is plotted against the local first derivative of the data. The straight line portion of this graph at large time, if it exists, yields simultaneously a constant (asymptotic value) plus an exponential, which can be subtracted from the given curve and the procedure repeated. The method permits judging the data in simple fashion for suitability of fit by exponentials simultaneously with the fitting itself. The method is formulated as an analytic version of Prony's method, which permits clarification of certain difficulties of the latter. Procedures are suggested for fitting exponentials with closely spaced half-lives, and for fitting functions other than exponential. The method is illustrated by application to data on the time course of respiratory uptake of a highly fat-soluble inert gas by human subjects.

Poggio, A.J., M.L. Van Blaricum, E.K. Miller, and R. Mittra, "Evaluation of a Processing Technique for Transient Data", IEEE Transactions on Antennas and Propagation, Vol. AP-26, No. 1, pp. 165-173, January 1978.

A data processing technique, namely Prony's method, is evaluated. The use of the algorithm for EMP and other transient problems is illustrated and the relationship between the waveform parameterization and the singularity expansion method (SEM) is described. The impulse response of a synthesizer network is obtained using Prony processing of the output of the double exponential excited circuit. Difficulties which arise in this type of processing such as rank deficiency, aliasing, and noise effects are considered and methods for alleviation such as filtering and rank-overspecification are introduced and evaluated. The overall status of waveform parameterization as applied to SEM is considered and suggestions for future research presented.

Prewitt, J.F. and K.S. Kunz, "SEM Pole Extraction: A Feasibility Study", Mission Research Corp., AMRC-N-81, March 1978.

This communication establishes typical experimental electromagnetic transient response measurement limitations, and applies these limitations to the natural mode decomposition of the response of a variety of scatterers due to EMP.

Price, H.J., "An Improved Prony Algorithm for Exponential Analysis", Mathematics Note 59, November 1978.

An improved Prony algorithm is described which produces a lower sum of squares of residuals than the usual Prony algorithm. Indeed, the new algorithm produces the lowest sum of squares of residuals possible. A simple numerical example is used to compare the new and old algorithms. Within the context of the new algorithm, a technique to constrain poles is also described.

Price, M.L., et al., "Electrical and Photon Tests of a Resonant Satellite Shape," IEEE Transactions on Nuclear Science, Vol. NS-25, No. 6, pp. 1358-1364, December 1978.

A resonant body has been tested and analyzed to determine its SGEMP response. Both electrical and photon excitation were used and their results compare favorably with predictions. A modal analysis of the experimental data using Prony's algorithm yielded the dominant frequencies, damping and excitation amplitudes. Within the accuracy of the measurements, the presence of the photon-induced electron cloud had no effect on the period or damping times of the RESMOD modes. The proximity of an electrical pulser and its 200-Ohm terminating resistor markedly decreased the damping time during electrical excitation.

Prony, R., "Essai Experimental et Analytique sur les Loix de la Dilatabilite de Fluides Elastiques et sur Celles de la Force Expansive de la Vaperu de l'Alkool, a Differentes Temperatures," J. l'Ecole Polytech. (Paris), Vol. 1, No. 2, pp. 24-76, 1795.

This paper has been abstracted in the Section 2.0 of this bibliography. This is Prony's original paper.

Quasi, A.H. and F.R. Spitznogle, "Time-Limited Signal Representation and Analysis by Using Non-Fourier Techniques," U.S. Navy Journal of Underwater Acoustics, Vol. 18, No. 3, pp. 397-402, July 1968.

Time-limited signals are represented and analyzed by using non-Fourier techniques. Results show that non-Fourier techniques can be superior to the conventional Fourier approach for the frequency analysis of time limited signals.

Salzer, H.E., "Some Extensions of Prony Approximation", Zeitschrift fur Angewandte Mathematik und Mechanik, 57, #4, pp. 269-271, 1977.

Prony's method of approximating a function  $f(s)$  by a sum of exponential terms  $\sum_{i=1}^n A_i e^{a_i x}$ , where both  $a_i$  and  $A_i$  are to be determined, is extended to oscillatory interpolation of any order, and also to the direct global approximation to the solution of the linear differential equation

$$\sum_{r=0}^k \theta_r(x) y^{(r)}(x) = f(x) \neq 0.$$

Sarkar, T.K., J. Nebat, and D.D. Weiner, "Suboptimal System Approximation/ Identification with Known Error", Air Force Weapons Laboratory, AFWL-TR-77-200, September 1977. Also Mathematics Note 49, 3 September 1977.

This paper presents a noniterative method for approximating empirical signals over  $[0, \infty]$  by a linear combination of exponentials. The technique results in a suboptimal approximation. Notably, the dependence of the suboptimal exponents  $s'_i$  on the integral square error  $\epsilon$  is such that

$\lim_{\epsilon \rightarrow 0} s'_i = s_i$ , the optimal exponents. The method may

also be used for system identification. It is especially useful when the system is modelled by a black box and one has access only to the input and output terminals of the system. A technique is demonstrated to find the multiple poles of a system along with the residues at the poles when the output of the system to a known input is given. Among the advantages of the method are its natural insensitivity to noise in the data and the explicit determination of the signal order. Representative computations are made of the poles from the transient response of a conducting pipe tested at the ATHAMAS-I EMP simulator.

Sarkar, T.K., V.K. Jain, J. Nebat, and D.D. Weiner, "A Comparison of the Pencil-of-Function Method with Prony's Method, Wiener Filters and Other Identification Techniques", Rochester Institute of Technology, Mathematics Note 54, December 1977.

The problem of interest is to identify the transfer function of a system by its poles and residues when the noise contaminated input and output are specified. The first aim of this paper is to illustrate that several different formulations for characterizing the impulse response of a system yield the same set of poles as is obtained in the case of a discrete Wiener filter. The second goal is to show how different formulations regularize the ill-posed system identification problem.



It is demonstrated that the Weiner filter is not always realizable as a causal rational function. When the order of the filter is specified a priori, the resulting filter may no longer be optimum. Finally, representative computations are made of the poles from the transient response of a conducting pipe tested at the ATHAMAS-I EMP simulator to demonstrate the stability, reliability, consistency, and accuracy of the results obtained by the pencil-of-function method.

Sarkar, T.K., D.D. Weiner, and V.K. Jain, "Some Mathematical Considerations in Dealing with the Inverse Problem", IEEE Transactions on Antennas and Propagation, Vol. AP-29, No. 2, pp. 373-379, March 1981.

Many problems of mathematical physics can be formulated in terms of the operator equation  $Ax = y$ , where  $A$  is an integro-differential operator. Given  $A$  and  $x$ , the solution for  $y$  is usually straightforward. However, the inverse problem which consists of the solution for  $x$  when given  $A$  and  $y$  is much more difficult. The following questions relative to the inverse problem are explored. 1) Does specification of the operator  $A$  determine the set  $\{y\}$  for which a solution  $x$  is possible? 2) Does the inverse problem always have a unique solution? 3) Do small perturbations of the forcing function  $y$  always result in small perturbations of the solution? 4) What are some of the considerations that enter into the choice of a solution technique for a specific problem? The concept of an ill-posed problem versus that of a well-posed problem is discussed. Specifically, the manner by which an ill-posed problem may be regularized to a well-posed problem is presented. The concepts are illustrated by several examples.

Scharf, L.L. and A.A. Beex, "Covariance Sequence Approximation for Parametric Spectrum Modelling", Technical Report August 80-1, Colorado State University, Ft. Collins, Colorado 80523.

Parametric methods of spectrum analysis are founded on finite-dimensional models for covariance sequences. Rational spectrum approximants for continuous spectra are based on autoregressive (AR), moving average (MA) or autoregressive moving average (ARMA) models for covariance sequences. Line spectrum approximants to discrete spectra are based on sinusoidal models for covariance sequences.

In this paper the authors make the point that a wide variety of spectrum types admit to modal analysis wherein the modes are characterized by amplitudes, frequencies, and damping factors. The associated modal decomposition is appropriate for both continuous and discrete components

of the spectrum. The domain of attraction for the decomposition includes ARMA sequences, harmonically- or nonharmonically-related sinusoids, damped sinusoids, white noise, and linear combinations of these.

The parametric spectrum analysis problem now becomes one of identifying mode parameters. This is achieved by solving two modified least-squares problems. Numerical results are presented to illustrate the identification of mode parameters and corresponding spectra from finite records of perfect and estimated covariance sequences. The results for sinusoids and sinusoids in white noise are interpreted in terms of inphase and quadrature effects attributable to the finite record length.

Scharf, L.L., A.A. Beex and T. von Reyn, "Modal Decomposition of Covariance Sequences for Parametric Spectrum Analysis", contributed paper, 1981 International Conference on Acoustics, Speech, and Signal Processing, Atlanta, March 30, 1981.

In this paper the authors make the point that a wide variety of spectrum types admit to modal analysis wherein the modes are characterized by amplitudes, frequencies, and damping factors. The associated modal decomposition is appropriate for both continuous and discrete components of the spectrum. The domain of attraction for the decomposition includes ARMA sequences, harmonically- or nonharmonically-related sinusoids, damped sinusoids, white noise, and linear combinations of these.

Numerical results are presented to illustrate the identification of mode parameters and corresponding spectra from finite records of perfect and estimated covariance sequences. The results for sinusoids and sinusoids in white noise are interpreted in terms of inphase and quadrature effects attributable to the finite record length.

Schaubert, D.H., "Application of Prony's Method to Time-Domain Reflectometer Data and Equivalent Circuit Synthesis", HDL-TR-1857, Harry Diamond Laboratories, June 1978. Also IEEE Transactions on Antennas and Propagation, Vol. AP-27, No. 2, pp. 180-184, March 1979.

The singularity expansion of time-domain reflectometer data has been computed and used to calculate lumped-element equivalent circuits for the impedance of some typical antennas. Prony's algorithm was used to obtain the poles of the antennas' terminal voltage waveform due to a step-like excitation. The residues, however, were calculated subject to the uniform-error norm instead of the least-squared-error norm. Although it requires much longer to

calculate, the uniform-error norm approximation is often orders of magnitude better than the least-squared-error approximation. The physical realizability of impedance functions obtained from experimental data has been investigated. Also, the procedures necessary to synthesize the lumped-element network have been evaluated. A simple partial-fraction expansion of a realizable impedance function representing an antenna was found to yield nonrealizable element values. A general network synthesis procedure, such as Brune's method, is required.

Schultz, E.R., "Estimation of Pulse Transfer Function Parameters by Quasilinearization", IEEE Transactions on Automatic Control, pp. 424-426, August 1968.

An iterative algorithm for the identification of single-input single-output linear stationary discrete system is developed using the method of quasilinearization. The resulting procedure is similar to mode 1 of the method of Steiglitz and McBride but has the advantage of the quadratic convergence property of quasilinearization. It is shown that this algorithm becomes mode 1 if the measured plant output is used in the calculations in place of the model output. Consequently, the two methods are extremely compatible and it is a simple matter to combine them in a single program, which generates its own initial estimate, has the wide range of convergence of mode 1, and possesses the quadratic convergence property of quasilinearization for final convergence to a solution. The method also permits the estimation of plant initial conditions in those cases where they must be considered. Results of a few numerical applications are discussed.

Scrivner, G.J., "Prony's Analysis in the Presence of Noise", Computer Sciences Corporation, Report II/3-C-235, October 1977. Also Mathematics Note 67, 7 October 1977.

This report studies a methodology, based on Prony's algorithm, for extracting complex frequencies and associated residues directly from transient response data. The methodology is not new but has been receiving increasing attention due to its potential usefulness in a variety of different applications in quite diverse scientific disciplines. Although this particular study was to evaluate the potential usefulness of Prony's method for analysis of System Generated Electromagnetic Pulse (SGEMP) experimental data, the results obtained are of a much broader nature. The report presents a classical derivation of the matrix equation which result if one uses Prony's algorithm in conjunction with a least-squares criterion.

The resulting matrix is studied with respect to the nature of its eigenvalue structure and an error analysis is developed which highlights the importance of this structure. The influence of noise on the eigenvalue structure and its impact on the determination of the number of poles in the data are examined. Optimization concepts relative to advantageous modification of the eigenvalue structure of the Prony matrix are qualitatively discussed. Difficulties associated with the two state application of the least-squares method in conventional Prony analysis are cited and an iterative method of removing this shortcoming described. Numerical calculations illustrating the majority of the above concepts are presented and discussed.

Siegman, E. and H.Y. Miller, "Unstable Optical Resonator Loss Calculations Using the Prony Method", Applied Optics, Vol. 9, No. 12, pp. 2729-2736, December 1970.

The eigenvalues for all the significant low-order resonant modes of an unstable optical resonator with circular mirrors are computed using an eigenvalue method called the Prony method. A general equivalence relation is also given, by means of which one can obtain the design parameters for a single-ended unstable resonator of the type usually employed in practical lasers, from the calculated or tabulated values for an equivalent symmetric or double-ended unstable resonator.

Simon, W., "A Method of Exponential Separation Applicable to Small Computers", Phys. Med. Biol., Vol. 15, No. 2, pp. 355-360, 1970.

A method has been developed for separation of the components of multicompartiment exponentials which operates on a small computer. It seems free of many of the difficulties of other methods and provides a reasonable test for the number of components.

Smith, W.D. and D.L. Lager, "Parametric Characterization of Random Processes Using Prony's Method", Lawrence Livermore Laboratory, UCRL-52673, 25 May 1979.

The compact parametric characterization of a random process is often valuable in problems involving signal classification and system identification. The method of Prony suggest two different approaches to obtaining such a characterization from a finite record of sampled data. The first approach is to extract the parameters of the estimated autocorrelation waveform. The second approach is to estimate the parameters of a white noise driven linear system that generates an output with the same characteristics as the given random process.

These two approaches are compared with other common signal processing methods. Also, the performance of each approach is evaluated theoretically and with computer-simulated and experimentally recorded data. Each approach is found to rapidly characterize both stationary and nonstationary random processes. The first approach is superior for the accurate parametric characterization of a stationary process. Alternately, the more precise second approach is best for monitoring changes with time in random process parameters.

Soussou, J.E., F. Moavenzadeh, and M.H. Gradowczyk, "Application of Prony Series to Linear Viscoelasticity", Transactions of the Society of Rheology, Vol. 14, No. 4, pp. 573-584, 1970.

A new method for description of viscoelastic functions with Prony series of exponentials is presented and compared with existing methods. This method places constraints on the coefficients of the series which guarantees smoothness of the functions and a discrete spectral representation. An optimization technique is used to obtain the solution. The resulting series permit the use of Whittaker's method of solution of integral equations which is specialized to the use of viscoelastic analysis, and an exact relationship is obtained between creep and relaxation functions.

Spitznogle, F.R. and A.H. Quazi, "Representation and Analysis of Time-Limited Signals Using a Complex Exponential Algorithm", Journal of the Acoustical Society of America, Vol. 47, No. 5, pp. 1150-1155, May 1970.

A complex exponential algorithm developed for the representation and analysis of time-limited signals is defined, and its evaluation with respect to conventional discrete Fourier techniques is discussed. It is shown that for a given length of a signal containing discrete frequency information, sampled at least to the Nyquist criterion, the complex exponential algorithm can often provide increased frequency resolution over standard Fourier techniques. It is also shown that the complex exponential algorithm provides an improved mechanism over Fourier techniques for interpolation between points in a sampled signal containing discrete frequency components in the presence of band-band noise. The effects of noise in the complex exponential technique and the computational difficulties associated with the present complex exponential algorithm are also discussed.

Texas Instruments Incorporated, "Annual Report - Representation and Analysis of Sonar Signals Volume I: Improvements in the Complex Exponential Signal Analysis Computational Algorithm," Texas Instruments, Inc., Dallas, Texas, 31 March 1970.

This volume of the 1969-70 Annual Report on Contract N00014-69-C-0315 describes the results of a study to reduce the computer memory and computational time required to obtain a complex exponential signal representation. As described in the report, substantial reductions in computational complexity have been achieved through improvements in the Prony method for solving the nonlinear system of equations associated with a complex exponential signal analysis. Projections of the real-time capability of a hard-wired complex exponential processor using state-of-the-art hardware are given. In addition, two iterative complex exponential algorithms based on the Newton-Raphson and gradient techniques for solving a nonlinear system of equations are described.

Trivett, David H., "A Modified Prony Method Approach to Echo-Reduction Measurements of Time-Limited Transient Signals," NRL Memorandum Report 4172, June 16, 1980, ADA 087211.

A modified Prony method is presented for measuring the steady-state echo reduction of acoustic panels. The method extrapolates the steady-state amplitudes from the transient portion of the signal allowing time-limited measurements. The method is applied to measurements of square panels 76 cm on an edge and 0.95 cm-thick steel and aluminum in the frequency range of 3 - 10 kHz. The signals were time-limited to  $200\mu\text{s}$  ( $0.6 - 2.0\lambda$ ) by the arrival of the diffracted signal from the panel edges. Results are compared with theoretical values and a program listing is included.

Van Blaricum, M.L. and R. Mittra, "A Technique for Extracting the Poles and Residues of a System Directly from its Transient Response," IEEE Transactions on Antennas and Propagation, Vol. AP-23, No. 6, pp. 777-781, November 1975, also Interaction Note 245, February 1975.

A novel approach for systematically deriving the complex poles and residues of a structure from a set of time-domain data is presented. The method is based on Prony's algorithm which involves the inversion of two matrices and a solution of the zeros of an nth degree polynomial, n being the number of desired poles. Two numerical examples are presented and several numerical advantages of this method are discussed.

Van Blaricum, M.L., "Techniques for Extracting the Complex Resonances of a System Directly from its Transient Response," Ph.D. Dissertation, Department of Electrical Engineering, University of Illinois, December 1975 also Interaction Note 301, December 1975.

The singularity expansion method is a techniques used for writing the transient response of a structure as a sum of exponentially damped sinusoids. In order to apply this method, it is first necessary to obtain the singularities of the system being studied. The conventional approach for determining the singularities of a system is based on an iterative search procedure that seeks the zeros of the system determinant in the complex frequency plane. The alternative approach of extracting the system singularities directly from the transient response function is discussed. The method developed is based on Prony's algorithm. Some basic problems which are associated with the use of Prony's algorithm are discussed and solutions are obtained.

It is demonstrated that Prony's method is applicable to systems with multiple as well as simple pole singularities. Two techniques are presented for systematically determining the number of poles contained in a transient response.

Van Blaricum, M.L. and R. Mittra, "PSPL: A Computer Program for Extracting the Poles and Residues from a Transient Signal Using Prony's Method", Electromagnetics Laboratory Scientific Report No. 7602, University of Illinois, Urbana, Illinois, March 1976.

This report describes the Fortran program PSPL, which is used to extract the poles and residues of a system from a set of discrete transient data using a least-squares version of Prony's method. The code has been written so that multiple as well as simple poles can be obtained if they are present in the data. The listing of the program given in this report is for use on the CDC-7600 computer but can be easily adapted to other machines.

Van Blaricum, M.L. and R. Mittra, "PEIGEN: A Computer Program for the Determination of the Number of Poles in a Transient Signal", Electromagnetics Laboratory Report No. 76-5, University of Illinois, Urbana, Illinois, June 1976.

This is a users program for the PEIGEN program which is used to determine the number of poles,  $N$  in a set of transient data assuming that the data are made up of a finite sum of exponentials. PEIGEN is written as a companion to the program PSPL, which is used to extract the  $N$  poles from the transient data once the value of  $N$  has been determined. The method uses the eigenvalue method.

Van Blaricum, M.L., "The Use of Prony's Method as an Analysis Tool Applied to Electrical Excitation of FltSatCom", Mission Research Corporation Report, MRC-R-301, January 1977.

This report discusses the use of the numerical algorithm known as Prony's method as a tool for the analysis of electrical transient experimental data. The data to which these techniques are applied as examples are results of electrical excitation tests of a structural mockup of the FltSatCom satellite.

Van Blaricum, M.L. and R. Mittra, "Problems and Solutions Associated with Prony's Method for Processing Transient Data", IEEE Transactions on Antennas and Propagation, Vol. AP-26, No. 1, pp. 174-182, January 1978.

Three difficulties associated with Prony's method are studied. These are the extension of the method to allow multiple poles, the development of techniques for determining the number of poles contained in the transient data, and the effects of noise in the data on the numerical procedures. Solutions to these difficulties are studied, and numerical samples are presented.



Weaver, J.J., D. Ng, and D. L. Lager, "Analytical Qualification of System Identification (Modal Analysis) Codes for Use in the Dynamic Testing of Nuclear Power Plant Structure", Lawrence Livermore Laboratory, UCID-18144, 2 January 1980.

This report describes the analytical evaluation of two particular system identification codes used at Lawrence Livermore Laboratory. Both codes are eigenparameter identification codes, however, one uses a time domain approach while the other a frequency domain approach. The evaluation was accomplished by analytically generating several time history signals in which the true modal parameters were known. These time histories ranged from widely spaced modes with spacing factors of 6 percent. These signals were then polluted with various levels of simulated measurement noise and the ability of our computer codes to extract the parameters from this noisy data was evaluated.

Weiss, L. and R.N. McDonough, "Prony's Method, Z-Transforms, and Pade Approximation", SIAM Review, Vol. 5, No. 2, pp. 145-149, April 1963.

The purpose in this note is to point out that the original form of Prony's method may also be regarded as a Pade approximation in the Z-transform domain.

Weygandt, C.N. and N.N. Puri, "Identification of a Linear System From Discrete Values of Input-Output Data", 1966 Proceeding 4th Annual Allerton Conference on Circuit and System Theory, po. 903-912, 1966.

In this paper the authors describe a method for determining the transfer function of a physical system having period input and output, both of which can be measured. The method depends upon taking  $2n$  equidistant samples of the input and the output within one period,  $n$  being the number of poles the system transfer function is assumed to have. These samples are used to compute  $(n+1)$  determinants and thereby the coefficients and the roots of an  $n$ th order polynomial. The roots of this  $n$ th order polynomial are further processed to obtain the poles of the transfer function. Once the location of poles is determined, a very simple computer routine determines the residues of these poles and thereby the zeros and the gain constant are determined. An essential feature of the scheme is that it determines the poles independent of the zeros. In practical examples the number of samples taken is much larger than  $2n$ , thereby generating a large number of  $n$ th order polynomials. The average of the roots of these polynomials are used to locate the poles.

White, O.R., "Inversion of the Limb-Darkening Equation Using the Prony Algorithm", The Astrophysical Journal, Vol. 152, pp. 217-223, April 1968.

King's interesting application of the Prony algorithm to the inversion of the limb-darkening equation is modified and generalized so that it can be applied rigorously to limb-darkening curves, spectral line profiles, and multiplet line intensities. Since King's formulation requires physically inaccessible observations, a change of variable is introduced in the limb-darkening integral to avoid this difficulty. Calculations on noisy data confirm the self-limiting property of this inversion method. An approximate method for constructing the smooth  $S(\tau)$  distribution from the slab solution is presented.

Wiscombe, W.J. and J. W. Evans, "Exponential-Sum Fitting of Radiative Transmission Functions", Journal of Computational Physics, 24, pp. 416-444, 1977.

Fitting transmission functions with exponential sums is the basis for a widely used approximation for calculating spectrally integrated radiative fluxes in planetary atmospheres, especially when both line absorption and scattering are important. The error in this method depends crucially on the accuracy of the fits, but unfortunately exponential-sum fitting is a classical ill-conditioned problem of numerical analysis. Previous techniques devised for exponential-sum fitting are often unsatisfactory in this application. This part presents a new method which sidesteps the ill-conditioning, guarantees convergence to the unique best least-squares fit, gives positive coefficients, and produces fits orders of magnitude more accurate than any which have so far been published. The method is demonstrated to be capable of recovering an exponential sum, given data sampled from that sum and rounded to as few as two decimal places. Sample fits are given for the Goody and Malkmus random band models and the Yamamoto H<sub>2</sub>O solar absorption data in order to illustrate the high accuracy of the method. The effect on the fits or errors in the transmission data is examined in some depth.

Wolf, J.K., "Decoding of Bose-Chaudhuri-Hocquenghem Codes and Prony's Method for Curve Fitting", IEEE Transactions on Information Theory, p. 608, October 1967.

The literature is filled with examples of a common set of equations which arise in two or more diverse applications. The purpose of this correspondence is to point out that such a situation has occurred in the two fields of

1) algebraic coding theory, and 2) curve fitting. Two aspects are of particular interest: the large time span which separates the fundamental works in these two areas, and that the methods of solution are identical.

Wynn, P., "A Note on the Fitting of Certain Types of Experimental Data", Statistica Neerlandica, Vol. 16, No. 2, pp. 143-150, 1962.

This paper is concerned with the problem of determining those values of the parameters  $Z, A_i, \alpha_i (i=1,2,\dots,h)$  which provide a best fit of the form

$$f^*(t) = Z + \sum_{i=1}^k A_i e^{-\alpha_i t}$$

to the observed values  $f_s = f(t_s) (s=1,2,\dots,n)$  in the sense

that the sum  $\sum_{i=1}^k \{f^*(t_s) - f(t_s)\}^2$  is a minimum.

Two iterative numerical techniques for the solution of this problem are discussed. The first, a variable gradient method, converges rapidly but involves a relatively large amount of computation; the second involves less computation but converges less rapidly. It is shown that by application of a suitable acceleration technique to the second method, its convergence is made as rapid as that of the first.

Yengst, W.C., "Approximation to a Specified Time Response," IRE Transactions on Circuit Theory, pp. 152-163, June 1962.

This paper presents a procedure by which specified data or a function of time  $h^*(t)$  can be approximated by trigonometric and/or exponential functions of time  $h(t)$  for which the Laplace transformations  $H(s)$  are known and can be expressed in rational fraction form. The procedure is based on fitting  $h(t)$  by an  $m$ th-order difference equation whose coefficients are determined by a least-squares technique. These coefficients are used directly to determine the poles of  $H(s)$ . The zeros of  $H(s)$  are established by using the prescribed data or function  $h^*(t)$  and the initial value theorem. The approximate function of time is obtained by taking the inverse Laplace transformation of  $H(s)$ . By this procedure not only is an approximation obtained for  $h^*(t)$  in the time domain, but its transform is also found in rational fraction form suitable for realization as a driving point or transfer function. Furthermore, the least-squares technique used in determining most or all of the unknown parameters in this procedure tends to minimize the effect of random errors or noise present in the specified data.

Youla, D.C., "The Identification of Linear Dynamical Systems from Time-Domain Measurements," Department of Electrical Engineering - Farmingdale, New York 11735, pp. 1-26, Appendix A1-A2, 1 September 1978.

The problem of identifying the impulse response of a lumped, linear time-invariant system from a discrete set of its samples, is essentially finite-dimensional in character. Prony's method is distinguished by the fact that it recognizes this finite-dimensionality from the outset and thereby succeeds in avoiding much of the numerical instability exhibited by algorithms that are designed to work in an infinite-dimensional setting. This paper attempts to generate some new insight into both the theoretical and practical implications of Prony's ideas.

Young, T.Y., "Representation and Analysis of Signals, Part XVI, Representation and Detection of Multiple-Epoch Signals," The Johns Hopkins University, Baltimore, Maryland, AS-405028, May 1963.

A multiple-epoch signal consists of several signals which for one reason or another, overlap one another. The purpose of this report is to design a procedure for the detection of the individual epochs of the overlapping signals and to represent them properly. The multiple-epoch signal may be corrupted with random Gaussian noise of zero mean. The individual signals are assumed to be representable by a set of exponential functions with acceptable error, and the exponents of this set of exponentials are assumed known. Any two adjacent epochs are assumed to be separated at least  $T_0$  seconds apart.

For the signal uncorrupted with noise a criterion based on the error energy is described which is of theoretical interest.

This criterion is useful for the detection of first epoch only. To detect the second epoch, the first signal is subtracted from the original signal by means of the "complementary" operator concept. For a noisy signal, a likelihood ratio criterion is proposed. The preliminary experimental results carried out on a digital computer justify the theoretical study.

Young, T.Y. and W.H. Huggins, "Representation and Analysis of Signals Part XIII: Discrete Orthonormal Exponentials," The Johns Hopkins University, Department of Electrical Engineering, Baltimore, Maryland, AS-405028, May 1963.

In dealing with the problem of signal analysis, one set of very useful component functions is the orthonormal exponential functions. Previous studies have concentrated mostly, on continuous exponentials with real exponents. With recent progress in high-speed digital computer, it is desirable to perform the signal analysis on digital computer. This requires the transformation of orthonormal exponentials from the continuous form into the sampled data form, that is, the z-transformation.

This paper deals with digitizing the orthonormal exponentials with complex exponents. The z-transforms of continuous orthonormal exponentials are not themselves orthogonal in z-domain. This is due to the fundamental property of sampling that a signal with a frequency spectrum higher than the sampling frequency is not exactly reproducible from its sampled data. For this reason, the Kautz process is introduced in the z-domain to define a set of discrete orthonormal exponentials. These discrete orthonormal exponentials approach the ordinary continuous orthonormal exponentials in the limit as the sampling interval  $T$  approaches zero.

Young, T.Y. and W.H. Huggins, "On the Representation of Electrocardiograms", IEEE Transactions on Bio-Medical Electronics, Vol. BME-10, No. 3, pp. 86-95, July 1963.

In dealing with the sophisticated statistical analysis of medical signals such as the electrocardiogram (ECG), the first problem one encounters is how to describe each ECG by a few numbers. This is the problem of efficient representation of signals, i.e., to approximate the signal with the smallest number of basis signals while preserving the accuracy of the approximation. This paper begins with a discussion of signal representation in general. The concept of signal space is introduced, which is very helpful in understanding the ideas of signal representation. This portion of the material is of tutorial nature.

Attention is then directed to a set of basis components which have found to be very efficient for ECG representation. These components are the so-called orthonormal exponential signals. An iterative process is developed which enables one to find a set of matched exponents for the representation of all ECG's. With six pairs of such exponentials, the average error of ECG representation (QRS and T waves only, leaving out P wave) is in the vicinity of five percent. Experimental results will be shown. Using this representation, further statistical analysis may be carried out with ease.

#### 4.0 PAPERS PRESENTED AT CONFERENCES, SYMPOSIA AND MEETINGS

Auton, J.R. and L.W. Pearson, "An Adaptive Filtering Algorithm for the Identification of SEM Poles," 1979 USNC/URSI Fall Meeting, Boulder, Colorado, November 1979.

Auton, J.R. and M.L. Van Blaricum, "On the Source of Parameter Bias in Prony's Method," 1980 NEM Conference, Disneyland Hotel, August 1980.

Auton, J.R., "An Unbiased Method for the Estimation of the SEM Parameters of an Electromagnetic System," 1980 NEM Conference, Disneyland Hotel, August 1980.

Auton, J.R. and M.L. Van Blaricum, "A Preprocessing Technique for Identification of Scatterer Resonances from Measured Transient Response," IEEE/AP-S Symposium, Los Angeles, CA June 16-19, 1981.

Beatty, L.G. and A.Z. Robinson, "Future Time Prediction of Transient Signals from Limited Time Measurements Using the Prony Method," 92nd Meeting: Acoustical Society of America, Journal of Acoustical Society of America, Vol. 60, Supplement No. 2, Fall 1976.

Brittingham, J.N., E.K. Miller, and J.L. Willows, "A Technique for Obtaining Simple Poles from Real-Frequency Information," Proceedings of National Conference on Electromagnetic Scattering, University of Illinois at Chicago Circle, pp. 106-109, 15-18 June 1976.

Brittingham, J.N., E.K. Miller, and J.L. Willows, "A Procedure to Find Simple Poles from Magnitude-Only Real Frequency Information," 1976 USNC/URSI Meeting, University of Massachusetts, Amherst, Massachusetts, 10-15 October 1976.

Brittingham, J.N., E.K. Miller, A.J. Poggio, and J.L. Willows, "Prony's Method for Frequency Domain," 1977 Mini-Symposium on Modal Analysis of Experimental Data, IEEE Albuquerque Joint Chapter, 21-22 March 1977.

Cho, K.S., C.A. Lin and J.T. Cordaro, "SEM Parameter Measurement," 1980 NEM Conference, Disneyland Hotel, August 5-7, 1980.

Chuang, C.W. and D.L. Moffatt, "Complex Natural Resonances of Radar Targets via Prony's Method," 1975 URSI Meeting, University of Illinois, Urbana, Illinois, pp. 67-68, 3-5 June 1975.

Chuang, C.W., D.L. Moffatt, and A.A. Ksienski, "Natural Resonance LOCI of Interacting Targets," 1977 IEEE AP-S/URSI Meeting, Stanford University, Stanford, California, 20-24 June 1977.

Cordaro, J.T., "Least Squares Prony's Method Applied to Pipe Data," 1977 Mini-Symposium on Modal Analysis of Experimental Data, IEEE Albuquerque Joint Chapter AP-S, S-MTT, G-EMC, also known as 1977 Spring FULMEN Meeting, 21-22 March 1977.

Cordaro, J.T., "Measured Poles from the Transient Response of a Cylinder," 1977 USNC/URSI-APS Symposium, Stanford University, Stanford, California, 20-24 June 1977.

Cordaro, J.T., "Some Pole Extraction Results Using Iterative Methods," 1978 National Radio Science Meeting, Boulder, Colorado, 9-13 January 1978.

Cordaro, J.T., "Pole Measurements for the Athamas I Pipe Test," 1978 Nuclear EMP Meeting, University of New Mexico Campus, IEEE, Albuquerque Joint Chapter, 6-8 June 1978.

Davis, W.A., "An Investigation of the Ill-Posed Nature of Prony's Method," Modeling and Simulation, Vol. 9, Proceedings of Ninth Annual Pittsburgh Conference, Instrument Society of America, Pittsburgh, pp. 889-903, April 1978.

Deadrick, F.J., et al., "Object Discrimination Via Pole Extraction From Transient Fields," Program of the 1975 URSI Meeting, University of Illinois, Urbana, Illinois, p. 67, 3-5 June 1975.

Deadrick, F.J. and A.J. Poggio, "Experimentally Determined Transient Responses and Natural Frequencies for Conducting Structures," 1978 USNC/URSI Meeting, University of Maryland, College Park, Maryland, p.14 15-19 May 1978.

Dudley, D.G., "Random Errors in Prony's Method," 1976 USNC/URSI Meeting, University of Massachusetts, Amherst, Massachusetts, 10-15 October 1976.

Dudley, D.G., "Improving Parameter Estimates in Transient Electromagnetic Modeling," 1978 USNC/URSI Meeting, Boulder, Colorado, 9-13 January 1978.

Fromme, D.A. et al., "Exploding-Wire Photon Testing of Skynet Satellite," IEEE Annual Conference on Nuclear and Space Radiating Effects," University of New Mexico, July 18-21, 1978.

Gavel, D.T., J.V. Candy and J.E. Zicher, "Parameter Estimation From Noisy Transient Electromagnetic Measurements," 1980 NEM Conference, Disneyland Hotel, Anaheim, CA, August 5-7, 1980.

Hudson, G.H., F.J. Deadrick, A.J. Poggio, and E.K. Miller, "Computational Difficulties in the Extraction of Poles from Transient Data," 1975 Annual Meeting, University of Colorado, Boulder, Colorado, 20-23 October 1975.

Hudson, H.G., "Observations on the Operation of the SEMPEX Code," 1977 Mini-Symposium on Modal Analysis of Experimental Data and 1977 Spring FULMEN Meeting, IEEE, Albuquerque Joint Chapter AP-S, S-MTT, G-EMC, 21-22 March 1977.

Lager, D.L., R.J. Lytle, and E.K. Miller, "Inversion of One-Dimensional Scattering Data Using Prony's Method," 1978 International IEEE/AP-S Symposium USNC/URSI Spring Meeting, Washington, D.C., 15-19 May 1978.

Markel, J.D., "The Prony Method and its Application to Speech Analysis," The Journal of the Acoustical Society of America, 80th Meeting of the Acoustical Society of America, Vol. 49, No. 1 (Part 1), pp. 105-106, 1971.



Miller, E.K., J.N. Brittingham, and J.L. Willows, "Identification of EM Spectrum by Known Pole Sets," 1977 IEEE AP-S/URSI Meeting, Stanford University, Stanford, California, 20-24 June 1977.

Miller, E.K., A. Aggarwal, and D. Lager, "Information Extraction Using Prony's Method," 1977 IEEE AP-S/URSI Meeting, Stanford University, Stanford, California, 20-24 June 1977.

Miller, E.K. and D.L. Lager, "Application of Prony's Method to Radiation-Pattern Imaging," 1978 National Radio Science Meeting, Boulder, Colorado, 9-13 January 1978.

Miller, E.K., "Prony's Method Revisited," 1978 International IEEE/AP-S Symposium, USNC/URSI Spring Meeting, Washington, D.C., 15-19 May 1978.

Miller, E.K., "A Study of Target Identification Using Poles," 1978 National Radio Science Meeting, USNC/URSI, Boulder, Colorado, 6-9 November, 1978.

Mitra, R. and M.L. Van Blaricum, "A Novel Technique for Extracting the SEM Poles and Residues of a System Directly from its Transient Response," 1974 Annual URSI Meeting, University of Colorado, Boulder, Colorado, 14-17 October 1974.

Pearson, L.W., M.L. Van Blaricum, and R. Mitra, "A New Method for Radar Target Recognition Based on the Singularity Expansion for the Target," 1975 IEEE International Radar Conference, Arlington, Virginia, pp. 452-456, 21-23 April 1975.

Pearson, L.W., "The Determination of the SEM Description of a System by Way of Measured Data," 1977 Mini-Symposium on Modal Analyses of Experimental Data, Albuquerque Joint Chapter AP-S, S-MTT, G-EMC, 1977 Spring FULMEN Meeting 21-22 March 1977.

Pearson, L.W. and D. Randall Roberson, "On the Extraction of the SEM Quantities for a Thin Wire from its Transient Response," 1978 National Radio Science Meeting in Boulder, Colorado, 9-13 January 1978.

Poggio, A.J. and J.N. Brittingham, "Complex Natural Frequency and Amplitude Extraction Using Rational Function Approximation of Spectral Data," 1976 AP-S Symposium, University of Massachusetts, Amherst, Massachusetts, 11-15 October 1976.

Price, H.J., "An Improved Prony Algorithm for Exponential Analysis," IEEE, EMC Symposium, San Diego, CA Oct. 9-11, 1979.

Price, H.J., "An Improved Algorithm for Representation of Frequency-Domain Data by Rational Functions," 1980 NEM Conference, Anaheim, CA August 5-7, 1980.

Price, M.L., et al., "Electrical and Photon Tests of a Resonant Satellite Shape," 1978 IEEE Annual Conference on Nuclear and Space Radiation Effects, University of New Mexico, Albuquerque, New Mexico, 18-21 July 1978.

Sarkar, T.K., "Suboptimal System Approximation/Identification by Complex Exponentials," 1977 Mini-Symposium on Model Analysis of Experimental Data, IEEE Albuquerque Joint Chapter AP-S, S-MTT, G-EMC, 1977 Spring FULMEN Meeting, 21-22 March 1977.

Sarkar, T.K., J. Nebat, and D.D. Weiner, "Suboptimal System Approximation/Identification by Complex Exponentials," 1977 IEEE AP-S/URSI Meeting, USNC/URSI, Boulder, Colorado, 6-9 November 1978.

Sarkar, T.K., "Comparison of Prony's Method (Covariance Method -CV) and Autocorrelation Method (AM)," 1978 National Radio Science Meeting, USNC/URSI, Boulder, Colorado, 6-9 November 1978.

Sarkar, T.K., "Is Prony's Method an Ill Posed Problem and Pencil-of-Function Method a Well Posed Problem in System Identification," URSI Meeting, Seattle, Washington, June 18-22, 1979.

Schaubert, D.H., "Application of Prony's Method to Time Domain Reflectometer Data and Equivalent Circuit Synthesis," 1978 USNC/URSI Meeting, Boulder, Colorado 9-13 January 1978.

Scrivner, G.J. and B.K. Singaraju, "Prony Analysis Utilizing Optimization Concepts and Iterative Improvement," 1977 Mini-Symposium on Modal Analysis of Experimental Data, IEEE, Albuquerque Joint Chapter AP-S, S-MTT, G-EMC, 1977 Spring FULMEN Meeting 21-22 March 1977.

Scrivner, G.J. and B.K. Singaraju, "An Iterative Technique for Extending Prony's Method to the Analysis of Noise Data," 1977 IEE AP-S/URSI Meeting, Stanford University, Stanford, California, 20-24 June 1977.

Scrivner, G.J., "First Order Error Analysis of Prony's Method," 1978 National Radio Science Meeting, Boulder, Colorado, 9-13 January 1978.

Scrivner, G.J., "The Noise Induced Statistical Behavior of the Eigenvalue Structure of the Prony Method," 1978 International IEEE/AP-S Symposium, USNC/URSI Spring Meeting, Washington, D.C., 15-19 May 1978.

Uberall, Herbert, et al., "Physical Basis of the Singularity Expansion Method in Terms of Creeping Waves (Theory and Experiment)," 1981 National Radio Science Meeting, Los Angeles, California, 15-19 June 1981.

Van Blaricum, M.L., L.W. Pearson, and R. Mittra, "An Efficient Scheme for Radar Target Recognition Based on the Complex Natural Resonances of the Target," 1975 AP-S Symposium, University of Illinois, Urbana, Illinois, 2-4 June 1975.

Van Blaricum, M.L. and R. Mittra, "Techniques for the Extension and Generalization of Prony's Method," 1975 Annual URSI Meeting, University of Colorado, Boulder, Colorado, 20-23 October 1975.

Van Blaricum, M.L. and R. Mittra, "Techniques for Applying Prony's Method to Nonideal Transient Signals," DNA EMP Interaction and Coupling Analysis and Validation Seminar, U.S. Air Force Academy, Colorado, 20-22 July 1976.

Van Blaricum, M.L. and D. Schaubert, "An Experimental Transient Transfer Function Via Prony's Method," 1976 USNC/URSI Meeting, University of Massachusetts, Amherst, Massachusetts, 10-14 October 1976.

Van Blaricum, M.L., "An Analysis of Existing Prony's Method Techniques," 1977 Mini Symposium on Model Analysis of Experimental Data, IEEE Joint Chapter AP-S, S-MTT, G-EMC, 1977 Spring FULMEN Meeting, 21-22 March 1977.

Van Blaricum, M.L., "A Review of Prony's Method Techniques for Parameter Estimation," RADC Spectrum Estimation Workshop, May 1978.

Van Blaricum, M.L., "Difficulties Present in Algorithms for Determining the Rank and Proper Poles with Prony's Method," RADC Spectrum Estimation Workshop, October 1979.

Van Blaricum, M.L. and J.R. Auton, "Identification of Scatterer Resonances by Reversing the Measured Response In Time," RSI Symposium, Los Angeles, California, July 16-19, 1981.

van Lint, V.A.J. and M.L. Van Blaricum, "Application of Prony Method to SGEMP Data Analysis," 1978 NEM Conference, Albuquerque, New Mexico, 6-8 June 1978.

Volakis, J., and L. Peters Jr., "Elimination of Undesired Natural Resonances for Improved Target Identification," URSI Symposium, Quebec, Canada, June 2-6, 1980.

## 5.0 BOOKS

Anderssen, R.S. and M.R. Osborne, Data Representation, University of Queensland Press, Queensland, Australia, pp. 62-76, pp. 77-93, pp.94-101, 1970.

Buckingham, R.A., Numerical Methods, Sir Isaac Pitman and Sons, London, pp. 329-333, 1957.

Felsen, L.B., Transient Electromagnetic Fields, Chapter 2, by R. Mittra, Springer-Verlag, New York, pp. 116-123, 1976.

Hamming, R.W., Numerical Methods for Scientists and Engineers, 2nd. Ed., McGraw-Hill, New York, pp. 620-627, 1973.

Hildebrand, F.B., Introduction to Numerical Analysis, McGraw-Hill, New York, pp. 378-382, 1956.

Jones, D.S., Methods in Electromagnetic Wave Propagation, Clarendon-Press, Oxford, 1979.

Kelly, L.G., Handbook of Numerical Methods and Applications, Addison-Wesley, Reading, Massachusetts, pp. 80-82, 1967.

Korn, G.A. and T.M. Korn, Mathematical Handbook for Scientists and Engineers, McGraw-Hill, New York, pp. 766-768.

Lanczos, C., Applied Analysis, Prentice Hall, Englewood Cliffs, New Jersey, pp. 272-280, 1956.

Markel, J.D. and A.H. Gray, Jr., Linear Predictions of Speech, Springer-Verlag, New York, Chapter 2, 1973.

Su, K.L., Time-Domain Synthesis of Linear Networks, Prentice Hall, Englewood Cliffs, New Jersey, pp. 23-26, 1971.

Willers, F.A., Practical Analysis, Dover Publications, Inc., Dover, New York, pp. 355-363.

Wittaker, E. and G. Robinson, The Calculus of Observations, Dover Publications, Inc., Dover, New York, pp. 369-371, 1967.