Mathematics Notes

Note 89

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Energy Norms and 2-Norms

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Abstract

This paper defines an energy norm or e-norm based on the energy delivered to some port in a system. It is expressible as a weighted 2-norm. Criteria are developed for the frequency spectrum of time-domain excitation waveforms to bound the 2-norms and e-norm at the port. Canonical incident waveforms are introduced to give convenient bounds.

I. Introduction

In characterizing system response to electromagnetic excitation it is useful to use norms because of the inequalities (bounds) which are an integral part of their definition. In addition, some norms can be related to damage/upset mechanisms such as energy, peak signal, etc. Various previous papers [1-5] deal with norms of time-domain waveforms. This paper extends our consideration of the 2-norms of such signals (which might be called the action-integral norm) to what can be called the energy norm or e-norm. This can be expressed as a weighted 2-norm and represents the energy delivered to some port of interest.

Fundamental to the norms in this paper is the relation between norms in frequency and time domains which are extended here. This also shows how the frequency spectrum of excitation waveforms must be controlled to achieve the correct norm values. Canonical incident waveforms for plane wave excitation are discussed from the point of view of bounding the response norms for a variety of possible excitations. Step-function excitation is particularly interesting leading to the concept of an effective volume to characterize the energy absorption at the port.

II. Generalized Parseval Theorem

The well-known Parseval theorem states [2]

$$\int_{-\infty}^{\infty} f^2(t)dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(j\omega)\tilde{f}(-j\omega)d\omega \tag{2.1}$$

$s = \Omega + j\omega \equiv$ complex frequency or Laplace-transform variable

$\sim \equiv$ Laplace transform (two sided) designator

In this form f(t) can even be complex, but normally we are concerned with real-valued time functions. Stated in the form of a 2-norm we have

$$\begin{aligned} \|f(t)\|_2 \equiv \left\{\int_{-\infty}^{\infty} f^2(t)dt\right\}^{1/2} &= \frac{1}{\sqrt{2\pi}}\left\{\int_{-\infty}^{\infty} \tilde{f}(j\omega)\tilde{f}(-j\omega)d\omega\right\}^{1/2} \\ &\equiv \frac{1}{\sqrt{2\pi}}\left\|\tilde{f}(j\omega)\right\|_2 \end{aligned} \tag{2.2}$$

where now f(t) is assumed real so that (conjugate symmetry)

$$\tilde{f}(-j\omega) = \tilde{f}^*(j\omega) \tag{2.3}$$

$* \equiv$ complex conjugate

This result expresses a concept that the energy in a time-domain signal is also computable in frequency domain.

As shown in [2] the Parseval theorem is a special case of

$$\int_{-\infty}^{\infty} f_1(t)\, f_2(t)dt = \frac{1}{2\pi j} \int_{Br} \tilde{f}_1(s)\tilde{f}_2(-s)ds \tag{2.4}$$

Br $\equiv$ Bromwich contour in strip of convergence in s plane

The strip of convergence is to the right of singularities of $\tilde{f}_1(s)$ and to the left of singularities of $\tilde{f}_2(-s)$ provided $f_1(t)$ and $f_2(t)$ are zero before some t (a real variable, taken here as time). Setting $f_1(t)$ and $f_2(t)$ equal to $f(t)$ gives

$$\int_{-\infty}^{\infty} f^2(t)dt = \frac{1}{2\pi j} \int_{Br} \tilde{f}(s)\tilde{f}(-s)ds \tag{2.5}$$

which is equivalent to (2.1) if the Bromwich contour is taken as the $j\omega$ axis. As used in [4] this form involving the integral of an analytic Kernel (specifically no conjugation) along a contour in the complex frequency plane allows one to deform the contour for convenience of evaluation of the integral.

III. Energy Norm

Suppose now that we interpret $f_1(t)$ as voltage and $f_2(t)$ as current at some port in a system of interest. With sign convention giving positive energy into the port and assuming a passive system with no initially stored energy we have

$$\int_{-\infty}^{t} V(t')\, I(t')\, dt' \geq 0 \text{ for all } t \tag{3.1}$$

as discussed in [2]. Applying (2.4) we have

$$\begin{aligned}
\int_{-\infty}^{t} V(t)\, I(t)\, dt &= \frac{1}{2\pi j} \int_{Br} \tilde{V}(s)\, \tilde{I}(-s) = \frac{1}{2\pi j} \int_{Br} \tilde{V}(-s)\, \tilde{I}(s) ds \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{V}(j\omega)\, \tilde{I}(-j\omega) d\omega = \frac{1}{2\pi j} \int_{-\infty}^{\infty} \mathrm{Re}\left[\, \tilde{V}(j\omega) \tilde{I}(-j\omega) \,\right] d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{V}(-j\omega)\, \tilde{I}(j\omega) d\omega
\end{aligned} \tag{3.2}$$

giving several alternate forms for evaluation in frequency domain.

Assuming that the port of interest is linear and time invariant, we can characterize it by an impedance or admittance

$$\begin{aligned}
\tilde{V}(s) &= \tilde{Z}(s) \tilde{I}(s) \\
\tilde{I}(s) &= \tilde{Y}(s) \tilde{V}(s) \\
\tilde{Z}(s) \tilde{Y}(s) &= 1
\end{aligned} \tag{3.3}$$

which in time domain is

$$V(t) = Z(t) \circ I(t)$$

$$I(t) = Y(t) \circ V(t)$$

$$Z(t)\circ = [Y(t)\circ]^{-1}$$

$$[Y(t)\circ]^{-1} = Z(t)\circ$$

$$Z(t) \circ Y(t)\circ = Y(t)\circ Z(t)\circ = \delta(t)\circ$$

$$\circ \equiv \text{convolution} \tag{3.4}$$

The requirement that the port be passive for every frequency leads to the concept of a p.r. function [7] which has properties

$$\tilde{Z}(s^*) = \tilde{Z}^*(s) \ , \ \tilde{Y}(s^*) = \tilde{Y}^*(s)$$

(conjugate symmetry implying real values for real s)

$$\mathrm{Re}[\tilde{Z}(s)] \geq 0 \quad , \quad \mathrm{Re}[\tilde{Y}(s)] \geq 0 \quad \text{for } \mathrm{Re}[s] \geq 0 \tag{3.5}$$

Such functions have the properties that there are no singularities in the right half plane (RHP), poles on the $j\omega$ axis must be simple with positive real residues, and that as $s \to \infty$ in the RHP they behave s^{υ} times a positive real constant with $-1 \leq \upsilon \leq 1$ (and for lumped element networks $\upsilon = -1, 0, 1$).

So the integrals in (3.2) can be restated in terms of either voltage or current with admittance or impedance as a kind of weighting function. Stating this in terms of a norm (the energy norm, or e-norm) we have

$$\begin{aligned} \|V(t), Y(t)\circ\|_e &\equiv \left\{ \int_{-\infty}^{\infty} V(t)\,[Y(t) \circ V(t)]dt \right\}^{1/2} \\ &= \|I(t), Z(t)\circ\|_e \equiv \left\{ \int_{-\infty}^{\infty} I(t)\,[Z(t) \circ I(t)]\,dt \right\}^{1/2} \\ &= \left\{ \int_{-\infty}^{\infty} V(t)\,I(t)dt \right\}^{1/2} \end{aligned} \tag{3.6}$$

Notationally we can think of this as a weighted norm with the convolution operator after the comma. Note that it is like a 2-norm in that V(t) and I(t) appear quadratically in the integrand and the square root is taken of the integral. Note that if the operator were $\delta(t)\circ$ (the identity) this would be precisely a 2-norm.

To see that this is a norm go to frequency domain as in (3.2) giving

$$\int_{-\infty}^{\infty} V(t)\,I(t)\,dt = \frac{1}{2\pi j}\int_{Br} \tilde{V}(s)\,\tilde{I}(-s)ds = \frac{1}{2\pi j}\int_{Br} \tilde{V}(-s)\,\tilde{I}(s)$$
$$= \frac{1}{2\pi j}\int_{Br} \tilde{V}(s)\,\tilde{Y}(-s)\tilde{V}(-s)ds = \frac{1}{2\pi j}\int_{Br} \tilde{V}(s)\,\tilde{Y}(s)\tilde{V}(-s)ds$$
$$= \frac{1}{2\pi j}\int_{Br} \tilde{I}(s)\,\tilde{Z}(-s)\tilde{I}(-s)ds = \frac{1}{2\pi j}\int_{Br} \tilde{I}(s)\,\tilde{Z}(s)\,\tilde{I}(-s)ds \tag{3.7}$$

Note now that the convolution in time domain has been replaced by multiplication in frequency domain. Further specializing to the $j\omega$ axis we have

$$\int_{-\infty}^{\infty} V(t)\,I(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\tilde{V}(j\omega)\right|^2 \mathrm{Re}\left[\tilde{Y}(j\omega)\right]d\omega$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\tilde{I}(j\omega)\right|^2 \mathrm{Re}\left[\tilde{Z}(j\omega)\right]d\omega \tag{3.8}$$

Note the use of the real part of the admittance and impedance since the integral must be real valued (as on the left). Now return to (3.5) showing that these real parts must be non-negative. Restricting these functions to have only isolated zeros of the real parts on the $j\omega$ axis then we can say that

$$\left\|\tilde{V}(j\omega), \tilde{Y}(j\omega)\right\|_e \equiv \left\{\int_{-\infty}^{\infty} \left|\tilde{V}(j\omega)\right|^2 \mathrm{Re}\left[\tilde{Y}(j\omega)\right]d\omega\right\}^{1/2}$$
$$= 0 \text{ iff } \tilde{V}(j\omega) \equiv 0$$

(except at isolated ωs) provided $\mathrm{Re}\left[\tilde{Y}(j\omega)\right] > 0$ (except at isolated ωs) (3.9)

$$\left\|\tilde{I}(j\omega), \tilde{Z}(j\omega)\right\|_e \equiv \left\{\int_{-\infty}^{\infty} \left|\tilde{I}(j\omega)\right|^2 \mathrm{Re}\left[\tilde{Z}(j\omega)\right]d\omega\right\}^{1/2}$$
$$= 0 \text{ iff } V(j\omega) \equiv 0$$

(except at isolated ωs) provided $\mathrm{Re}\left[\tilde{Z}(j\omega)\right] > 0$ (except at isolated ωs)

Thus provided the weight is positive (except at isolated ωs) we can take this as a norm, specifically a weighted 2-norm. In fact, we have the alternate representations

$$\left\|\tilde{V}(j\omega), \tilde{Y}(j\omega)\right\|_e = \left\|\tilde{V}(j\omega)\left[\mathrm{Re}\left[\tilde{Y}(j\omega)\right]^{1/2}\right]\right\|_2$$
$$\left\|\tilde{I}(j\omega), \tilde{Z}(j\omega)\right\|_e = \left\|\tilde{I}(j\omega)\left[\mathrm{Re}\left[\tilde{Z}(j\omega)\right]^{1/2}\right]\right\|_2 \tag{3.10}$$

Since clearly scalars come out of a weighted 2-norm as magnitude, this leaves only the triangle inequality to establish the norm property. This follows in voltage form as

$$
\begin{aligned}
\left\|\tilde{V}_2(j\omega)+\tilde{V}_2(j\omega),\,\tilde{Y}(j\omega)\right\|_e &= \left\|\tilde{V}_1(j\omega)\left[\mathrm{Re}\left[\tilde{Y}(j\omega)\right]\right]^{1/2}+\tilde{V}_2(j\omega)\left[\mathrm{Re}\left[\tilde{Y}(j\omega)\right]\right]^{1/2}\right\|_2 \\
&\le \left\|\tilde{V}_1(j\omega)\left[\mathrm{Re}\left[\tilde{Y}(j\omega)\right]\right]^{1/2}\right\|_2+\left\|\tilde{V}_2(j\omega)\left[\mathrm{Re}\left[\tilde{Y}(j\omega)\right]\right]^{1/2}\right\|_2 \\
&= \left\|\tilde{V}_1(j\omega),\,\tilde{Y}(j\omega)\right\|_e+\left\|\tilde{V}_2(j\omega),\,\tilde{Y}(j\omega)\right\|_e
\end{aligned}
\tag{3.11}
$$

and similarly for the current form. Thus the e-norm satisfies all the properties of a norm with suitable restriction on $\tilde{Y}(s)$ and $\tilde{Z}(s)$. While our discussion of the norm properties has been in the frequency-domain form, since by (3.6) and (3.8) there is a strict equality with the time domain form as

$$
\begin{aligned}
\left\|V(t),\,Y(t)\circ\right\|_e &= \frac{1}{\sqrt{2\pi}}\left\|\tilde{V}(j\omega),\,\tilde{Y}(j\omega)\right\|_e \\
\left\|I(t),\,Z(t)\circ\right\|_e &= \frac{1}{\sqrt{2\pi}}\left\|\tilde{I}(j\omega),\,\tilde{Z}(j\omega)\right\|_e
\end{aligned}
\tag{3.12}
$$

then all the norm properties apply to the time-domain form as well.

IV. Bounds on e-Norm in Terms of 2-Norm

As in [5] we have the Hölder inequality now applied to functions of ω over the range $-\infty$ to ∞ as

$$
\begin{gathered}
\left|\int_{-\infty}^{\infty} g_1(\omega) g_2(\omega)\, d\omega\right| \le \|g_1(\omega)\|_{p_1} \|g_2(\omega)\|_{p_2} \\
1 = \frac{1}{p_1} + \frac{1}{p_2} \\
p_1 \ge 1 \,,\ p_2 \ge 1
\end{gathered}
\tag{4.1}
$$

where limiting cases of 1 and ∞ for p are permissible. For the above formula ω can just as well be t.

Consider first the integrals involving V and I as in (3.2). Interpreting p_1 and p_2 as both 2 we have

$$
\begin{gathered}
0 \le \int_{-\infty}^{\infty} V(t)\, I(t) dt \le \|V(t)\|_2 \|I(t)\|_2 \\
0 \le \int_{-\infty}^{\infty} \tilde{V}(j\omega) \tilde{I}(-j\omega) d\omega \le \|\tilde{V}(j\omega)\|_2 \|\tilde{I}(j\omega)\|_2
\end{gathered}
\tag{4.2}
$$

Stated in norm form

$$
\begin{gathered}
\|V(t), Y(t) \circ\|_e = \|I(t), Z(t) \circ\|_e \le \left\{\|V(t)\|_2 \|I(t)\|_2\right\}^{1/2} \\
\|\tilde{V}(j\omega), \tilde{Y}(j\omega)\|_e = \|\tilde{I}(j\omega), \tilde{Z}(j\omega)\|_e \le \left\{\|\tilde{V}(j\omega)\|_2 \|\tilde{I}(j\omega)\|_2\right\}^{1/2}
\end{gathered}
\tag{4.3}
$$

indicating that the e-norm is bounded by the geometric mean of the 2-norms of the voltage and current. As in (3.12) the frequency and time forms above are all simply related.

Now look at the integrals as in (3.9). We have the simple bounds

$$
\begin{gathered}
\|\tilde{V}(j\omega), \tilde{Y}(j\omega)\|_e \le \|\tilde{V}(j\omega)\|_2 \left\{\sup_{\omega} \mathrm{Re}\left[\tilde{Y}(j\omega)\right]\right\}^{1/2} \\
\|\tilde{I}(j\omega), \tilde{Y}(j\omega)\|_e \le \|\tilde{I}(j\omega)\|_2 \left\{\sup_{\omega} \mathrm{Re}\left[\tilde{Z}(j\omega)\right]^{1/2}\right\}
\end{gathered}
\tag{4.4}
$$

provided the real parts of the admittance and/or impedance are bounded on the $j\omega$ axis.

V. Choosing Incident Spectrum to Bound e-Norm and 2-Norm of Response

Let there be transfer functions $\tilde{T}_V(s)$ for voltage and $\tilde{T}_I(s)$ for current from some incident or source waveform $F^{(inc)}(t)$. This incident waveform may represent an incident field (electric or magnetic), a voltage or current source in the system, etc. Then we have norms

$$\|V(t)\|_2 = \frac{1}{\sqrt{2\pi}}\|\tilde{V}(j\omega)\|_2 = \frac{1}{\sqrt{2\pi}}\left\|\tilde{T}_V(j\omega)\tilde{F}^{(inc)}(j\omega)\right\|_2$$
$$\|I(t)\|_2 = \frac{1}{\sqrt{2\pi}}\|\tilde{I}(j\omega)\|_2 = \frac{1}{\sqrt{2\pi}}\left\|\tilde{T}_I(j\omega)\tilde{F}^{(inc)}(j\omega)\right\|_2 \tag{5.1}$$

$$\|V(t),Y(t)\circ\|_e = \|I(t),Z(t)\circ\|_e = \left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{V}(j\omega)\,\tilde{I}(-j\omega)d\omega\right\}^{1/2}$$
$$= \left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{T}_V(j\omega)\tilde{T}_I(-j\omega)\left|\tilde{F}^{(inc)}(j\omega)\right|^2 d\omega\right\}^{1/2}$$
$$= \left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\tilde{T}_V(j\omega)\tilde{F}^{(inc)}(j\omega)\right|^2 \mathrm{Re}\left[\tilde{Y}(j\omega)\right]d\omega\right\}^{1/2}$$
$$= \left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\tilde{T}_I(j\omega)\tilde{F}^{(inc)}(j\omega)\right|^2 \mathrm{Re}\left[\tilde{Z}(j\omega)\right]d\omega\right\}^{1/2}$$

Next consider some other waveform $F(t)$ which is directly compared to $F^{(inc)}(t)$ in that it represents the same physical parameter with the only difference being the temporal or frequency dependence. Specifically, we require that the transfer functions remain unchanged. Suppose we constrain

$$\left|\tilde{F}(j\omega)\right| \geq \left|\tilde{F}^{(inc)}(j\omega)\right| \text{ for all } \omega \tag{5.2}$$

Then we have the norm inequalities

$$\left\|\tilde{T}_V(j\omega)\tilde{F}(j\omega)\right\|_2 \geq \left\|\tilde{T}_V(j\omega)\tilde{F}^{(inc)}(j\omega)\right\|_2$$
$$\left\|\tilde{T}_I(j\omega)\tilde{F}(j\omega)\right\|_2 \geq \left\|\tilde{T}_I(j\omega)\tilde{F}^{(inc)}(j\omega)\right\|_2$$

$$\left\|\tilde{T}_V(j\omega)\tilde{F}(j\omega),\tilde{Y}(j\omega)\right\|_e \geq \left\|\tilde{T}_V(j\omega)\tilde{F}^{(inc)}(j\omega),\tilde{Y}(j\omega)\right\|_e \tag{5.3}$$
$$\left\|\tilde{T}_I(j\omega)\tilde{F}(j\omega),\tilde{Z}(j\omega)\right\|_e \geq \left\|\tilde{T}_I(j\omega)\tilde{F}^{(inc)}(j\omega),\tilde{Z}(j\omega)\right\|_e$$

These are tight bounds in that equality is achieved in some cases. Specifically if equality holds for some ω in (5.2), then if the transfer function is zero for frequencies except very near this ω the inequalities in (5.3) become equalities.

This concept can be extended to a set of incident waveforms $F_n^{(inc)}$ for n = 1,2,...,N. Require

$$\left|\tilde{F}(j\omega)\right| \geq \max_{1 \leq n \leq N} \left|\tilde{F}_n^{(inc)}(j\omega)\right| \text{ for all } \omega \tag{5.4}$$

Then the inequalities in (5.3) apply for all $F_n^{(inc)}$. At least in 2-norm and e-norm senses, then this provides a way for establishing some criteria waveform $F(t)$ which covers all of some set of environmental waveforms $F^{(inc)}(t)$. The bound must hold for all <u>frequencies</u> if it is to apply to a generally unknown transfer function which can emphasize any frequencies at all.

VI. Norms for Response to Canonical Incident Waves

Consider that our system of interest is excited by an incident plane wave of the form

$$\vec{E}^{(inc)}(\vec{r},t) = E_o \vec{1}_e f^{(inc)}\left(t - \frac{\vec{1}_1 \cdot \vec{r}}{c}\right)$$

$$\vec{H}^{(inc)}(\vec{r},t) = H_o \vec{1}_h f^{(inc)}\left(t - \frac{\vec{1}_1 \cdot \vec{r}}{c}\right)$$

$$E_o = Z_o H_o$$

$$Z_o = \sqrt{\frac{\mu_o}{\varepsilon_o}} \equiv \text{characteristic impedance of free space}$$

$$c = \frac{1}{\sqrt{\mu_o \varepsilon_o}} \equiv \text{speed of light} \tag{6.1}$$

$$\vec{1}_1 \equiv \text{direction of incidence}$$

$$\vec{1}_e \equiv \text{polarization (assumed constant)}$$

$$\vec{1}_1 \times \vec{1}_e = \vec{1}_h \,,\ \vec{1}_e \times \vec{1}_h = \vec{1}_1 \,,\ \vec{1}_h \times \vec{1}_1 = \vec{1}_e$$

The transfer functions discussed in the previous section are now taken with respect to the above excitation such that we have

$$\tilde{V}(s) \equiv \tilde{T}_V(s) E_o \tilde{f}^{(inc)}(s)$$

$$\tilde{I}(s) \equiv \tilde{T}_I(s) H_o \tilde{f}^{(inc)}(s) \tag{6.2}$$

In this form the transfer functions both have dimensions of meters and are functions of not only s, but $\vec{1}_1$ and $\vec{1}_e$ (or equivalently $\vec{1}_1$ and $\vec{1}_h$) as well. If desired, one can think of these transfer functions as the response to an incident field of 1 V/m or 1 A/m.

Looking at the energy norm we can define

$$\operatorname{Re}\left[\tilde{A}_a(j\omega)\right] \equiv \operatorname{Re}\left[\tilde{T}_V(j\omega)\tilde{T}_I(-j\omega)\right] = \operatorname{Re}\left[\tilde{T}_V(-j\omega)\tilde{T}_I(j\omega)\right]$$

$$\equiv \text{absorption area or cross section} \tag{6.3}$$

Note that one can also define a complex version of this if one wishes as

$$\tilde{A}_a(s) \equiv \tilde{T}_V(s)\tilde{T}_I(-s)$$
$$= Z_o\tilde{Y}(-s)\tilde{T}_V(s)\tilde{T}_V(-s)$$
$$= \frac{\tilde{Z}(s)}{Z_o}\tilde{T}_I(s)\tilde{T}_I(-s) \tag{6.4}$$

with the real part on the $j\omega$ axis corresponding to (6.3). Note that one could equally define $\tilde{A}_a(s)$ with all signs on s reversed in (6.4). Now since our port is assumed passive we have

$$\operatorname{Re}\left[\tilde{A}_a(j\omega)\right] \geq 0 \text{ for all } \omega \tag{6.5}$$

and unless the port is purely reactive this will, in general, be positive. Applying this to the energy norm we have

$$\left\| V(t), Y(t)o \right\|_e = \left\| I(t), Z(t)o \right\|_e$$
$$= \left\{ \frac{E_oH_o}{2\pi} \int_{-\infty}^{\infty} \tilde{A}_a(j\omega) \left| \tilde{f}^{(inc)}(j\omega) \right|^2 d\omega \right\}^{1/2}$$
$$= \left\{ \frac{E_oH_o}{2\pi} \int_{-\infty}^{\infty} \operatorname{Re}\left[\tilde{A}_a(j\omega)\right] \left| \tilde{f}^{(inc)}(j\omega) \right|^2 d\omega \right\}^{1/2} \tag{6.6}$$

Let our incident wave be a step function as

$$\vec{E}_s^{(inc)}(\vec{r}, t) = E_o\vec{1}_e u\left(t - \frac{\vec{1}_1 \cdot \vec{r}}{c}\right)$$
$$\vec{H}_s^{inc}(\vec{r}, t) = H_o\vec{1}_h u\left(t - \frac{\vec{1}_1 \cdot \vec{r}}{c}\right) \tag{6.7}$$

Note that this wave has a constant energy density after the wavefront as

$$U = \frac{1}{2}\varepsilon_o E_o^2 + \frac{1}{2}\mu_o H_o^2 = \frac{E_oH_o}{c} \tag{6.8}$$

For the step response (6.6) becomes

$$\left\| V_s(t), Y(t)o \right\|_e = \left\| I_s(t), Z(t)o \right\|_e$$

$$= \left\{ \frac{E_o H_o}{2\pi} \int_{-\infty}^{\infty} \tilde{A}_a(j\omega)\omega^{-2} d\omega \right\}^{1/2}$$

$$= U^{1/2} V_e^{1/2}$$

$$V_e = \frac{c}{2\pi} \int_{-\infty}^{\infty} \tilde{A}_a(j\omega)\,\omega^{-2} d\omega = \frac{c}{\pi} \int_{0}^{\infty} \mathrm{Re}\left[A_a(j\omega)\right] \omega^{-2} d\omega \qquad (6.9)$$

$\equiv$ effective volume

This effective volume is a function of $\vec{1}_1$ and $\vec{1}_e$ and relates how much of the electromagnetic-field energy is delivered to the port. This is similar to the equivalent volume of a sensor [9] except that this latter parameter is based on the quasi-static characteristics of an optimally designed sensor. Here V_e is somewhat more general. Note that by changing ω to wavelength the integral in (6.9) is an integral over wavelength similar to the formula for the total cross section $\mathrm{Re}\left[\tilde{A}_t(j\omega)\right]$ in [8]. Of course, since

$$\mathrm{Re}\left[\tilde{A}_t(j\omega)\right] \geq \mathrm{Re}\left[\tilde{A}_a(j\omega)\right] \geq 0 \qquad (6.10)$$

then such an integral can bound V_e. However, note that here V_e is but one port of perhaps many in a system and A_a then may only account for a fraction of the absorbed power.

For step-function excitation we also have the 2-norms

$$\left\| V_s(t) \right\|_2 = \frac{E_o}{\sqrt{2\pi}} \left\| \frac{\tilde{T}_V(j\omega)}{j\omega} \right\|_2$$

$$\left\| I_s(t) \right\|_2 = \frac{H_o}{\sqrt{2\pi}} \left\| \frac{\tilde{T}_I(j\omega)}{j\omega} \right\|_2 \qquad (6.11)$$

Using the bound in (4.3) we have

$$0 \leq V_e \leq \frac{c}{2\pi} \left\| \frac{\tilde{T}_V(j\omega)}{j\omega} \right\|_2 \left\| \frac{\tilde{T}_I(j\omega)}{j\omega} \right\|_2 \qquad (6.12)$$

Note that if the port has a constant resistance, say R, for $\tilde{Z}(s)$, then $\tilde{T}_V$ and $\tilde{T}_I$ are the same except for a real constant multiplier simplifying the above results and making (6.12) an equality.

One reason for considering a step-function response is its utility for bounding some other common responses. One often considers canonical waveforms such as [8]

$$f_1(t) = \left[e^{-\alpha t} + e^{-\beta t}\right]u(t)$$

$$f_2(t) = \left\{e^{-\alpha t} + e^{\beta t}\right\}^{-1} \tag{6.13}$$

$$\alpha > 0 \quad , \quad \beta > 0$$

with two-sided Laplace transforms

$$\tilde{f}_1(s) = \frac{1}{s+\alpha} - \frac{1}{s+\beta} = \frac{\beta - \alpha}{(s+\alpha)(s+\beta)}$$

$$\tilde{f}_s(s) = \frac{\pi}{\alpha+\beta} \csc\left[\frac{\pi}{\alpha+\beta}(s+\beta)\right] \tag{6.14}$$

Here α^{-1} is a characteristic time for the rise and β^{-1} is a characteristic time for the fall. We have in time domain

$$0 \le f_1(t) < 1 \quad , \quad 0 < f_2(t) < 1 \text{ for all } t \tag{6.15}$$

Compared to the unit step the peak is less, but this is not the important point. On the $j\omega$ axis we have (without going into details)

$$\left|\tilde{f}_1(j\omega)\right| < \left|\frac{1}{j\omega}\right| \quad , \quad \left|\tilde{f}_2(j\omega)\right| < \left|\frac{1}{j\omega}\right| \text{ for all } \omega \tag{6.16}$$

Applying (5.2) we then have that the e-norms and 2-norms for response to f_1 and f_2 are bounded by the corresponding step response norms. Thus the results of (6.9) in terms of effective volume bound the response to waveforms in (6.13) for all positive α and β.

Besides a step response which essentially weights the transfer functions by $1/(j\omega)$ one might define other canonical examples which weight the frequencies differently. For example, we might take an impulsive wave as

$$\vec{E}_\delta^{inc}(\vec{r}\ ,t) = E_o \tau_1 \vec{1}_e \delta\left(t - \frac{\vec{1}_1 \cdot \vec{r}}{c}\right)$$

$$\vec{H}_\delta^{inc}(\vec{r}\ ,\ t) = H_o \tau_1 \vec{1}_h \delta\left(t - \frac{\vec{1}_1 \cdot \vec{r}}{c}\right) \tag{6.17}$$

where τ_1 is a time to account for the area of the pulse. Then we have the norms of the δ response as

$$\left\| V_\delta(t)\ ,\ Y(t)\ o \right\|_e = \left\| I_\delta(t)\ ,\ Y(t) o \right\|_e$$

$$= \tau_1 \left\{ \frac{E_o H_o}{2\pi} \int_{-\infty}^{\infty} \tilde{T}_V\ (j\omega)\, T_I(-\ j\omega) d\omega \right\}^{1/2}$$

$$\left\| V_\delta(t) \right\|_2 = \frac{E_o \tau_1}{\sqrt{2\pi}} \left\| \tilde{T}_V(j\omega) \right\|_2 \tag{6.18}$$

$$\left\| I_\delta(t) \right\|_2 = \frac{H_o \tau_1}{\sqrt{2\pi}} \left\| \tilde{T}_I(j\omega) \right\|_2$$

Compared to the step response this emphasizes the high frequencies. One will need the transfer functions to be small enough at high frequencies that the above integrals converge.

Another choice is a ramp wave as

$$\vec{E}_r^{inc}(\vec{r}\ ,\ t) = \frac{E_o}{\tau_2} \vec{1}_e \left[t - \frac{\vec{1}_1 \cdot \vec{r}}{c} \right] u\left(t - \frac{\vec{1}_1 \cdot \vec{r}}{c}\right)$$

$$\vec{H}_r^{inc}(\vec{r}\ ,\ t) = \frac{H_o}{\tau_2} \vec{1}_h \left[t - \frac{\vec{1}_1 \cdot \vec{r}}{c} \right] u\left(t - \frac{\vec{1}_1 \cdot \vec{r}}{c}\right) \tag{6.19}$$

where τ_2 is a time to account for the slope of the waveform. The norms of ramp response are then

$$\left\| V_r(t)\ ,\ Y(t)\ o \right\|_e = \left\| I_r(t)\ ,\ Z(t) o \right\|_e$$

$$= \frac{1}{\tau_2} \left\{ \frac{E_o H_o}{2\pi} \int_{-\infty}^{\infty} \tilde{T}_V\ (j\omega)\, \tilde{T}_I(-\ j\omega) d\omega \right\}^{1/2} \tag{6.20}$$

$$\left\| V_r(t) \right\|_2 = \frac{E_o}{\tau_2 \sqrt{2\pi}} \left\| \frac{\tilde{T}_V(j\omega)}{(j\omega)^2} \right\|_2$$

$$\left\| I_r(t) \right\|_2 = \frac{H_o}{\tau_2 \sqrt{2\pi}} \left\| \frac{\tilde{T}_I(j\omega)}{(j\omega)^2} \right\|_2$$

Compared to the step response this emphasizes the low frequencies. One will need the transfer functions at low frequencies to be sufficiently small that the above integrals converge.

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