Mathematics Notes

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A Simple Method of Generating a Basis of Solutions for Linear Partial Differential Equations

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Abstract

A simple procedure is given for constructing a complete basis of solutions to a given linear constant coefficient partial differential equation. A variety of examples are presented including eigenvalues problems such as the Helmholtz equation.

I. INTRODUCTION

If u is a smooth function that satisfies a linear constant coefficient partial differential equation (PDE) then the derivatives of u are related to each other in a linear fashion. This observation leads to a simple constructive procedure for generating a basis of solutions for the PDE.

For example, suppose that $u_x = u_t$ where the subscripts denote differentiation. If we let $u_{i,j} = \partial^{i+j}u/\partial x^i \partial t^j$ then $u_{1,0} = u_{0,1}$. Less trivially, the PDE also implies that $u_{xx} = u_{tx} = u_{tt}$. That is, $u_{2,0} = u_{1,1} = u_{0,2}$. Similarly, $u_{i,0} = u_{i-1,1} = \cdots = u_{0,i}$.

Expanding u in a Taylor series in x and t gives

$$\begin{aligned} u(x,t) &= u_{0,0} + u_{1,0}x + u_{0,1}t + u_{2,0}\frac{x^2}{2} + u_{1,1}xt + u_{0,2}\frac{t^2}{2} + \cdots \\ &= u_{0,0} + u_{1,0}(x+t) + u_{2,0}(\frac{x^2}{2} + xt + \frac{t^2}{2}) + \cdots \\ &= u_{0,0} + u_{1,0}(x+t) + u_{2,0}\frac{(x+t)^2}{2!} + \cdots + u_{i,0}\frac{(x+t)^i}{i!} + \cdots \qquad (1) \end{aligned}$$

Since the PDE does not relate $u_{i,0}$ to $u_{j,0}$ for $i \neq j$, the expansion (1) can not be reduced further. For this reason the values of $u_{i,0}$ for $i = 0, 1, \ldots$ are free (modulo any boundary conditions) and the functions

$$\psi_i \equiv \frac{(x+t)^i}{i!} \qquad (2)$$

form a basis of solutions. This basis is complete in the sense that its span includes any solution which can be expressed as a convergent power series (see [1] for a discussion of this restriction). Also note that the basis functions are exactly what we expect in terms of waves moving to the left: it is well known that if u satisfies $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t}$ then u is a function of $x + t$ only: $u(x,t) = u(x+t)$. In this case, $u(x+t) = \sum \frac{\partial^i u}{\partial x^i}\frac{(x+t)^i}{i!}$ which is just (1).

In effect, the above procedure is just an extension of the proof of the Cauchy-Kowalewski Theorem [2], [3] in which prescribed values of the x derivatives of u are used to reconstruct the function via the PDE. For this reason, we will call a basis that is generated in this way a Cauchy-Kowalewski basis.

In the following, we apply this procedure to a variety of linear PDEs. The restrictions that the PDE places on the power series coefficients allows us to rearrange the power series into a linear combination of basis solutions.

Application of this method to Laplace's equation (see below) yields the harmonic polynomials for two-dimensional problems and the spherical harmonics for three-dimensional problems. For other problems, this procedure generates solution bases that are nonseparable in nature [4], [5] [6], [7], [8]; such bases are often useful in solving boundary value problems on domains with complex geometries [9], [10].

In Section 2, the main results of this paper are developed for the Helmholtz equation in two dimensions. Because this is an eigenvalue PDE problem, the resulting basis solutions are given as power series rather than finite polynomials. However, we show that these solutions have a simple representation in terms of spherical Bessel functions of the first kind. These basis solutions are easily generated by the recursions given below and were used in [9] to find upper and lower bounds on the eigenvalues of the Laplacian over rhombic domains.

More generally, the procedures given in [11] for finding upper and lower bounds on the eigenvalues of elliptic operators can be implemented using the basis functions described in this paper.

In Section 3, the method is applied to three problems in higher dimensions: the Helmholtz equation in three dimensions and the anisotropic wave and heat equations.

Section 4 shows the connection between the symmetry operators of the two-dimensional Helmholtz equation [12] and the basis solutions of Section 2. Section 5 contains our conclusions.

We close this introduction with a look at Laplace's equation in two dimensions. Suppose that u satisfies

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 , \tag{3}$$

and has the power series expansion

$$u(x,y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} u_{ij} \, \frac{x^i}{i!} \, \frac{y^j}{j!} \quad , \tag{4}$$

where

$$u_{ij} = \frac{\partial^{i+j} u}{\partial x^i \partial y^j} \qquad \text{at} \ \ (x,y) = (0,0) \ . \tag{5}$$

By (3),

$$u_{2,0} = \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} = -u_{0,2} \,. \tag{6}$$

Thus the quadratic terms $u_{2,0}\frac{x^2}{2} + u_{0,2}\frac{y^2}{2}$ can be written as $u_{2,0}\left(\frac{x^2}{2} - \frac{y^2}{2}\right)$. Since no other restrictions apply to $u_{2,0}$, the function $\frac{x^2}{2} - \frac{y^2}{2}$ must satisfy Laplace's equation (as is easily seen by differentiating). This gives us the basis solution

$$\psi_{2,0} \equiv \frac{x^2}{2!} - \frac{y^2}{2!} \,. \tag{7}$$

Similarly, $u_{3,0} = -u_{1,2}$ so that

$$\begin{aligned} u_{3,0}\frac{x^3}{3!} + u_{1,2}\frac{xy^2}{2!} &= u_{3,0}\left(\frac{x^3}{3!} - \frac{xy^2}{2!}\right) \\ &= u_{3,0}\psi_{3,0}. \end{aligned}$$

These functions can also be obtained by using the extended separation of variables technique in [13]. From $u_{2,1} = -u_{0,3}$ we get the solution $\psi_{2,1} = \left(\frac{x^2 y}{2} - \frac{y^3}{3!}\right)$.

In general, we obtain the basis solutions

$$\psi_{2n+k,\ell} = \sum_{i=0}^{n} \frac{(-1)^{n-i} x^{2i+k} y^{2(n-i)+\ell}}{(2i+k)!(2(n-i)+\ell)!} \tag{8}$$

where $k = 0$ or 1, $\ell = 0$ or 1, and $0 \le n < \infty$.

These solutions are just the well known harmonic polynomials! For example,

$$\psi_{2,0} = \frac{x^2}{2} - \frac{y^2}{2} = \frac{r^2 \cos 2\theta}{2}$$

and

$$\psi_{2,1} = \frac{x^2 y}{2} - \frac{y^3}{3!} = \frac{r^3 \sin 3\theta}{3!} \,.$$

In the same way we can show that the Cauchy-Kowalewski basis for Laplace's equation in three dimensions simply consists of the spherical harmonics (see p. 192 in [14]).

Thus special solutions to particular equations can be obtained by a general procedure.

II. THE TWO-DIMENSIONAL HELMHOLTZ EQUATION

Assume that u can be expanded in a power series (4) which is absolutely convergent for $x^2 + y^2 < r^2$. If we further assume that u satisfies the Helmholtz equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\lambda^2 u, \tag{9}$$

then only the lower order coefficients u_{i0} and u_{i1} are needed since the higher order coefficients, u_{ij}, with $j \geq 2$ can be written in terms of these lower order coefficients. For example, by (9)

$$u_{02} = -\lambda^2 u_{00} - u_{20}. \tag{10}$$

Similarly,

$$\begin{aligned} u_{04} &= -\lambda^2 u_{02} - u_{22} \\ &= -\lambda^2 (-\lambda^2 u_{00} - u_{20}) - (-\lambda^2 u_{20} - u_{40}) \\ &= \lambda^4 u_{00} + 2\lambda^2 u_{20} + u_{40}\ . \end{aligned} \tag{11}$$

Since the series (4) is assumed to be absolutely convergent, we may rearrange its terms to eliminate the coefficients u_{ij} for $j > 1$. This gives our main result.

<u>*Theorem 1:*</u> If u satisfies (4) and (9) then for $x^2 + y^2 < r^2$,

$$u(x,y) = \sum_{n=0}^{\infty} H_{n,0}(x,y) u_{n,0} + \sum_{n=0}^{\infty} H_{n,1}(x,y) u_{n,1} \tag{12}$$

where the Helmholtz basis functions $H_{n,j}$ satisfy

$$\frac{\partial^2 H_{n,j}}{\partial x^2} + \frac{\partial^2 H_{n,j}}{\partial y^2} = -\lambda^2 H_{n,j}, \tag{13}$$

and are given by

$$H_{2n+i,j}(x,y) = \sum_{m=0}^{n} \frac{x^{2(n-m)+i}}{(2(n-m)+i)!} \left\{ \sum_{M=m}^{\infty} (-1)^M \binom{M}{m} \lambda^{2(M-m)} \frac{y^{2M+j}}{(2M+j)!} \right\} \tag{14}$$

for $i = 0$ or 1 and $j = 0$ or 1.

Proof: Write (9) as

$$\frac{\partial^2}{\partial y^2}u = -\left(\lambda^2 + \frac{\partial^2}{\partial x^2}\right)u. \tag{15}$$

From this we may express higher y derivatives in terms of lower derivatives by using a binomial expansion of $\left(\frac{\partial^2}{\partial y^2}\right)^M$:

$$\begin{aligned} u_{i,2M+J} &= \frac{\partial^i}{\partial x^i}\frac{\partial^{2M+J}}{\partial y^{2M+J}}u \qquad \text{for } J=0 \text{ or } 1 \\ &= \frac{\partial^i}{\partial x^i}\frac{\partial^J}{\partial y^J}\left(\frac{\partial^2}{\partial y^2}\right)^M u \\ &= \frac{\partial^i}{\partial x^i}\frac{\partial^J}{\partial y^J}\left(-\lambda^2 - \frac{\partial^2}{\partial x^2}\right)^M u \\ &= \frac{\partial^i}{\partial x^i}\frac{\partial^J}{\partial y^J}\sum_{m=0}^M(-1)^M\begin{pmatrix} M \\ m \end{pmatrix}\lambda^{2(M-m)}\frac{\partial^{2m}}{\partial x^{2m}}u \qquad (16) \\ &= \sum_{m=0}^M(-1)^M\begin{pmatrix} M \\ m \end{pmatrix}\lambda^{2(M-m)}\frac{\partial^{i+2m}}{\partial x^{i+2m}}\frac{\partial^J}{\partial y^J}u \\ &= \sum_{m=0}^M(-1)^M\begin{pmatrix} M \\ m \end{pmatrix}\lambda^{2(M-m)}u_{i+2m,J}\,. \end{aligned}$$

The power series of u can be written as

$$\begin{aligned} u(x,y) &= \sum_{i=0}^\infty\sum_{j=0}^\infty u_{i,j}\frac{x^i}{i!}\frac{y^i}{j!} \\ &= \sum_{I=0}^1\sum_{J=0}^1\sum_{P=0}^\infty\sum_{M=0}^\infty u_{2P+I,2M+J}\frac{x^{2P+I}}{(2P+I)!}\frac{y^{2M+J}}{(2M+J)!} \end{aligned} \tag{17}$$

where we have used the change of indices $i = 2P + I$ and $j = 2M + J$ so as to separate the odd and even powers of x and y.

Substituting (16) into (17) allows us to eliminate the higher order y coefficients in (17),

$$u(x,y) = \sum_{I=0}^1\sum_{J=0}^1\sum_{P=0}^\infty\sum_{M=0}^\infty\sum_{m=0}^M(-1)^M\begin{pmatrix} M \\ m \end{pmatrix}\lambda^{2(M-m)}\frac{x^{2P+I}}{(2P+I)!}\frac{y^{2M+J}}{(2M+J)!}u_{2P+I+2m,J}. \tag{18}$$

In order to bring together all the terms with the same value of $2P + I + 2m$, we use the relations

$$\sum_{M=0}^{\infty}\sum_{m=0}^{M} = \sum_{m=0}^{\infty}\sum_{M=m}^{\infty} \quad \text{and} \quad \sum_{P=0}^{\infty}\sum_{m=0}^{\infty} = \sum_{n=0}^{\infty}\sum_{m=0}^{n} .$$

where $n = P + m$ in the second series relation. This gives

$$\begin{aligned} u(x,y) &= \sum_{I=0}^{1}\sum_{J=0}^{1}\sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\frac{x^{2(n-m)+I}}{(2(n-m)+I)!}\sum_{M=m}^{\infty}(-1)^M\binom{M}{m}\lambda^{2(M-m)}\frac{y^{2M+J}}{(2M+J)!}\right\}u_{2n+I,J} \\ &= \sum_{I=0}^{1}\sum_{J=0}^{1} H_{2n+I,J}(x,y)u_{2n+I,J}. \end{aligned}$$

This proves (12) and (14) under the substitution of (i,j) for (I,J). Differentiating (14) shows that $H_{n,j}$ satisfies the Helmholtz equation (13). ∎

Efficient evaluation of the basis functions $H_{n,j}$ is important if the Helmholtz series (12) is to be used in numerical computations. The first few basis functions are given by,

$$H_{00}(x,y) = \cos\lambda y, \qquad H_{01}(x,y) = \frac{\sin\lambda y}{\lambda}, \tag{19}$$

and

$$H_{10}(x,y) = x\cos\lambda y, \qquad H_{11}(x,y) = \frac{x\sin\lambda y}{\lambda}. \tag{20}$$

For the next two basis functions, it is still not too hard to recognize the series in (14) as the product of polynomials and trigonometric functions:

$$\begin{aligned} H_{20}(x,y) &= \frac{x^2}{2}\cos\lambda y - y\frac{\sin\lambda y}{2\lambda} \\ H_{21}(x,y) &= \frac{x^2}{2}\frac{\sin\lambda y}{\lambda} + \frac{y\cos\lambda y}{2\lambda^2} - \frac{\sin\lambda y}{2\lambda^3}, \end{aligned}$$

but for the higher orders, we must proceed in a different manner. Fortunately, there is a simple method of evaluating the $H_{n,j}$ based on spherical Bessel functions of the first kind, j_n, where

$$\begin{aligned} j_n(z) &= \frac{z^n}{1\cdot3\cdot5\cdots(2n+1)}\left\{1 - \frac{\frac{1}{2}z^2}{1!(2n+3)} + \frac{(\frac{1}{2}z^2)^2}{2!(2n+3)\cdot(2n+5)} - \cdots\right\} \\ &= n!2^n z^n \sum_{k=0}^{\infty}\binom{n+k}{n}\frac{(-z^2)^k}{(2(n+k)+1)!}. \end{aligned} \tag{21}$$

In order to see the connection with $H_{n,j}$, write (14) as a finite sum

$$H_{2n+i,j}(x,y) = \sum_{m=0}^{n} \frac{x^{2(n-m)+i}}{(2(n-m)+i)!} g_{2m+j}(y) \tag{22}$$

where

$$g_{2m+j}(y) = \sum_{M=m}^{\infty} (-1)^M \binom{M}{m} \lambda^{2(M-m)} \frac{y^{2M+j}}{(2m+j)!} \quad , \tag{23}$$

for $j = 0$ or 1.

Lemma 2: Let $z = \lambda y$ then

$$g_{2m+1}(y) = \frac{(-1)^m z^{m+1} j_m(z)}{m! 2^m \lambda^{2m+1}} \qquad m = 0, 1, \ldots \tag{24}$$

$$g_{2m}(y) = \frac{(-1)^m z^{m+1} j_{m-1}(z)}{m! 2^m \lambda^{2m}} \qquad m = 1, 2, \ldots \tag{25}$$

Proof: Use (21) and (23)to get (24) and (25). ∎

Using Lemma 2 and the well known recursion [15]

$$j_{m+1}(z) = \left(\frac{2m+1}{z}\right) j_m(z) - j_{m-1}(z), \tag{26}$$

we obtain a recursion for the functions g_{2m+j}:

Lemma 3: For g_{2m+j} defined by (23),

$$\begin{aligned} g_{2m+3}(y) &= \frac{-(2m+1)}{2\lambda^2(m+1)} g_{2m+1}(y) - \frac{y^2}{4\lambda^2 m(m+1)} g_{2m-1}(y) \qquad m = 1, 2, \ldots \\ g_{2m+4}(y) &= \frac{-(2m+1)}{2\lambda^2(m+2)} g_{2m+2}(y) - \frac{y^2}{4\lambda^2(m+1)(m+2)} g_{2m}(y) \qquad m = 0, 1, \ldots \end{aligned} \tag{27}$$

Proof: Use (24) and (25) in (26) with $z = \lambda y$. ∎

The first few functions in this series are given by

$$g_0(y) = \cos \lambda y \quad , \quad g_1(y) = \frac{\sin \lambda y}{\lambda}$$

$$g_2(y) = \frac{-y \sin \lambda y}{2\lambda} \quad , \quad g_3(y) = \frac{y \cos \lambda y}{2\lambda^2} - \frac{\sin \lambda y}{2\lambda^3}$$

$$g_4(y) = \frac{-y^2}{8\lambda^2} \cos \lambda y + \frac{y^2}{8\lambda^3} \sin \lambda y$$

$$g_5(y) = \frac{-3y}{8\lambda^4} \cos \lambda y + \left[\frac{3}{8\lambda^5} - \frac{y^2}{8\lambda^3}\right] \sin \lambda y .$$

The recursions in Lemma 3 and the finite series (22) give us a simple method for evaluating the Helmholtz basis functions, $H_{n,j}$.

In general, $g_{2n}(y)$ is even in y while $g_{2n+1}(y)$ is odd. From this we see that $H_{2n,0}$ is even in both x and y, while $H_{2n+1,0}$ is odd in x and even in y. Similarly, $H_{2n,1}$ is even in x and odd in y while $H_{2n+1,1}$ is odd in x and y.

From (23) we find that

$$\begin{aligned} \frac{\partial}{\partial y} g_{2n+1} &= g_{2n} \\ \frac{\partial^2}{\partial y^2} g_{2n+1} &= -\lambda^2 g_{2n+1} - g_{2n-1} \,. \end{aligned} \tag{28}$$

These relations can be used to show that the basis functions obtained by the above procedure are related to those obtained by the method of "extended separation of variables" in [13]. Also note that since the Helmholtz equation in (9) is symmetric in x and y, we could have chosen to eliminate the higher order x derivatives rather than the higher order y derivatives.

III. HIGHER DIMENSIONAL PARTIAL DIFFERENTIAL EQUATIONS

The Cauchy-Kowalewski expansion method can be applied to any linear partial differential equation with constant coefficients in any number of variables. In this section, we apply this procedure to PDEs in three and four variables. In the interest of brevity, most of the details in the derivations will be suppressed.

THE ANISOTROPIC HELMHOLTZ EQUATION

Let u satisfy the anisotropic Helmholtz equation in three dimensions:

$$u_{zz} = au_{xx} + bu_{yy} + cu \,. \tag{29}$$

No assumptions are made as to the sign of the coefficients a, b and c or even whether they are real or complex.

We assume that u can be expanded in a power series

$$u(x,y,z) = \sum_{I=0}^{\infty}\sum_{J=0}^{\infty}\sum_{K=0}^{\infty} u_{I,J,K}\frac{x^I}{I!}\frac{y^J}{J!}\frac{z^K}{K!} \tag{30}$$

where

$$u_{I,J,K} = \frac{\partial^I}{\partial x^I}\frac{\partial^J}{\partial y^J}\frac{\partial^K u}{\partial z^K}\,.$$

By using a binomial expansion and (29) we obtain

$$\begin{aligned}\left[\frac{\partial^2}{\partial z^2}\right]^{\hat{K}} u &= \left[a\frac{\partial^2}{\partial x^2} + b\frac{\partial^2}{\partial y^2} + c\right]^{\hat{K}} u \\ &= \sum_{k_1=0}^{\hat{K}}\sum_{k_2=0}^{\hat{K}-k_1} \binom{\hat{K}}{k_1}\binom{\hat{K}-k_1}{k_2}\left(a\frac{\partial^2}{\partial x^2}\right)^{k_1}\left(b\frac{\partial^2}{\partial y^2}\right)^{k_2} c^{\hat{K}-k_1-k_2} u\,. \end{aligned} \qquad (31)$$

For $N_3 = 0$ or 1, this gives

$$u_{I,J,2\hat{K}+N_3} = \sum_{k_1=0}^{\hat{K}}\sum_{k_2=0}^{\hat{K}-k_1} \binom{\hat{K}}{k_1}\binom{\hat{K}-k_1}{k_2} a^{k_1} b^{k_2} c^{\hat{K}-k_1-k_2} u_{I+2k_1,J+2k_2,N_3}\,. \qquad (32)$$

With the following change of indices

$$\begin{aligned} I &= 2\hat{I} + N_1 \\ J &= 2\hat{J} + N_2 \\ K &= 2\hat{K} + N_3 \\ \hat{K} &= k_1 + k_2 + k_3 \\ \hat{J} &= j - k_2 \\ \hat{K} &= k - k_3 \\ \hat{I} &= i - k_1, \end{aligned} \qquad (33)$$

we may rewrite (30) as

$$\begin{aligned} u = {} & \textstyle\sum_{N_i=0}^{1}\sum_{\hat{I}=0}^{\infty}\sum_{\hat{J}=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}\frac{x^I}{I!}\frac{y^J}{J!}\frac{z^K}{K!} \\ & \binom{k_1+k_2+k_3}{k_1}\binom{k_2+k_3}{k_2} a^{k_1} b^{k_2} c^{k_3} u_{2\hat{I}+N_1+2k_1,2\hat{J}+N_2+2k_2,N_3}. \end{aligned} \qquad (34)$$

The summation relationships

$$\sum_{\hat{I}=0}^{\infty}\sum_{k_1=0}^{\infty} = \sum_{i=0}^{\infty}\sum_{k_1=0}^{i} \quad\text{and}\quad \sum_{\hat{J}=0}^{\infty}\sum_{k_2=0}^{\infty} = \sum_{j=0}^{\infty}\sum_{k_2=0}^{j} \qquad (35)$$

can be used to rewrite (34) as

$$u(x,y,z) = \sum_{N_1=0}^{1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left\{\sum_{k_1=0}^{i}\sum_{k_2=0}^{j}\sum_{k_3=0}^{\infty} a^{k_1} b^{k_2} c^{k_3}\right.$$

$$\frac{x^{2(i-k_1)+N_1}}{[2(i-k_1)+N_1]!}\quad\frac{y^{2(j-k_2)+N_2}}{[2(j-k_2)+N_2]!}\quad\frac{z^{2(k_1+k_2+k_3)+N_3}}{[2(k_1+k_2+k_3)+N_3]!} \tag{36}$$

$$\left.\begin{pmatrix} k_1+k_2+k_3 \\ k_1 \end{pmatrix}\begin{pmatrix} k_2+k_3 \\ k_2 \end{pmatrix}\right\} u_{2i+N_1,2j+N_2,N_3}\ .$$

Let $\psi_{2i+N,2j+N_2,N_3}$ be the expression in brackets in (36) and let

$$g_{2k_1,N_1,2k_2+N_2,N_3} = \sum_{k_3=0}^{\infty} a^{k_1} b^{k_2} c^{k_3}\begin{pmatrix} k_1+k_2+k_3 \\ k_1 \end{pmatrix}\begin{pmatrix} k_2+k_3 \\ k_2 \end{pmatrix} \tag{37}$$

$$\frac{z^{2(k_1+k_2+k_3)+N_3}}{[2(k_1+k_2+k_3)+N_3]!}\ .$$

Then

$$\psi_{2i+N_1,2j+N_2,N_3} = \sum_{k_1=0}^{i}\sum_{k_2=0}^{j}\frac{x^{2(i-k_1)+N_1}}{[2(i-k)+N_1]!}\frac{y^{2(j-k_2)+N_2}}{[2(j-k_2)+N_2]!} \tag{38}$$

$$g_{2k_1+N_1,2k_2+N_2,N_3}\ .$$

The two dimensional isotropic case is recovered by setting $a=0, b=-1, c=-\lambda^2, k_1=0, N_1=0$ in (37). For this case we find that

$$g_{0,2k_2+N_2,N_3} = \sum_{k_3=0}^{\infty}(-1)^{k_2+k_3}\lambda^{2k_3}\begin{pmatrix} k_2+k_3 \\ k_2 \end{pmatrix}\frac{z^{2(k_2+k_3)+N_3}}{[2(k_2+k_3)+N_3]!}$$

$$= \sum_{k=k_2}^{\infty}(-1)^{k}\lambda^{2(k-k_2)}\begin{pmatrix} k \\ k_2 \end{pmatrix}\frac{z^{2k+N_3}}{(2k+N_3)!} \tag{39}$$

with $k = k_2 + k_3$. As expected, this is the same result as that we obtained for the two-dimensional Helmholtz problem.

For the isotropic three-dimensional case, $a = b = -1, c = -\lambda^2$, the g functions are given by

$$g_{2k_1+N_1,2k_2+N_2,N_3} = \sum_{k=k_1+k_2}^{\infty}(-1)^{k}\lambda^{2[k-(k_1+k_2)]}\begin{pmatrix} k \\ k_1 \end{pmatrix}\begin{pmatrix} k-k_1 \\ k_2 \end{pmatrix}\frac{z^{2k+N_3}}{(2k+N_3)!}. \tag{40}$$

The lowest order example, $N_3 = N_2 = N_1 = k_1 = k_2 = 0$ gives

$$g_{0,0,0} = \sum_{k=0}^{\infty}(-1)^k \lambda^{2k} \frac{z^{2k}}{(2k)!} = \cos \lambda z \tag{41}$$

and for the $N_i = 0$ and $k_1 = k_2 = 1$, we get

$$g_{2,2,0} = \sum_{k=2}^{\infty}(-1)^k \frac{(\lambda z)^{2k}}{(2k)!} \frac{k(k-1)}{\lambda^4}. \tag{42}$$

If we use the relationship

$$\begin{pmatrix} k \\ k_1 \end{pmatrix} \begin{pmatrix} k - k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} k \\ k_1 + k_2 \end{pmatrix} \begin{pmatrix} k_1 + k_2 \\ k_2 \end{pmatrix}, \tag{43}$$

the isotropic g functions can be rewritten in a simpler form:

$$g_{2k_1+N_1,2k_2+N_2,N_3} = \begin{pmatrix} k_1 + k_2 \\ k_2 \end{pmatrix} \sum_{k=n}^{\infty}(-1)^k \lambda^{2(k-n)} \begin{pmatrix} k \\ n \end{pmatrix} \frac{z^{2k+N_3}}{(2k+N_3)!} \tag{44}$$

$$= \begin{pmatrix} k_1 + k_2 \\ k_2 \end{pmatrix} g_{2(k_1+k_2)+N_3}(z) . \tag{45}$$

To summarize the three-dimensional isotropic case, the expanded solution to (29) with $a = b = -1, c = -\lambda^2$ is

$$u(x, y, z) = \sum_{N_1=0}^{1} \sum_{N_2=0}^{1} \sum_{N_3=0}^{1} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} u_{2i+N_1,2j+N_2,N_3} \psi_{2i+N_1,2j+N_2,N_3} \tag{46}$$

where the basis functions are

$$\psi_{2i+N_1,2j+N_2,N_3} = \sum_{k_1=0}^{i} \sum_{k_2=0}^{j} \frac{x^{2(i-k_1)+N_1}}{[2(i-k_1)+N_1]!} \frac{y^{2(j-k_2)+N_2}}{[2(j-k_2)+N_2]!} g_{2(k_1+k_2)+N_3}(z, \lambda) \begin{pmatrix} k_1 + k_2 \\ k_2 \end{pmatrix} \tag{47}$$

with

$$g_{2k+N}(z, \lambda) = \sum_{m=k}^{\infty}(-1)^m \begin{pmatrix} m \\ k \end{pmatrix} \lambda^{2(m-k)} \frac{z^{2m+N}}{(2m+N)!} \tag{48}$$

and $N = 0$ or 1. Thus the results of the previous section concerning the two dimensional isotropic functions g_n of the previous section can be applied to the three-dimensional Helmholtz problem as well.

THREE-DIMENSIONAL ANISOTROPIC WAVE EQUATION

The three-dimensional wave equation

$$u_{tt} = au_{xx} + bu_{yy} + cu_{zz} \tag{49}$$

is of interest in the area of electromagnetic wave propagation. Let

$$u_{I,J,K,L} = \frac{\partial^I}{\partial x^I}\frac{\partial^J}{\partial y^J}\frac{\partial^K}{\partial z^K}\frac{\partial^L u}{\partial t^L}\,.$$

By using the wave equation and the binomial theorem with $N_4 = 0$ or 1, we find

$$\begin{aligned} u_{I,J,K,2\hat{L}+N_4} &= \frac{\partial^I}{\partial x^I}\frac{\partial^J}{\partial y^J}\frac{\partial^K}{\partial z^K}\frac{\partial^{N_4}}{\partial t^{N_4}}\left(\frac{\partial^2}{\partial t^2}\right)^{\hat{L}} u \\ &= \frac{\partial^I}{\partial x^I}\frac{\partial^J}{\partial y^J}\frac{\partial^K}{\partial z^K}\frac{\partial^{N_4}}{\partial t^{N_4}}\left(a\frac{\partial^2}{\partial x^2} + b\frac{\partial^2}{\partial y^2} + c\frac{\partial^2}{\partial z^2}\right)^{\hat{L}} u \\ &= \sum_{l_1=0}^{\hat{L}}\sum_{l_2=0}^{\hat{L}-l_1}\begin{pmatrix}\hat{L}\\ l_1\end{pmatrix}\begin{pmatrix}\hat{L}-l_1\\ l_2\end{pmatrix} a^{l_1}b^{l_2}c^{\hat{L}-(l_1+l_2)} \\ &\quad u_{I+2l_1,J+2l_2,K+2(\hat{L}-l_1-l_2),N_4}\,. \end{aligned} \tag{50}$$

This gives

$$\begin{aligned} u = & \sum_{N_4=0}^{1}\sum_{I=0}^{\infty}\sum_{J=0}^{\infty}\sum_{K=0}^{\infty}\sum_{\hat{L}=0}^{\infty}\sum_{l_1=0}^{\hat{L}}\sum_{l_2=0}^{\hat{L}-l_1}\begin{pmatrix}\hat{L}\\ l_1\end{pmatrix}\begin{pmatrix}\hat{L}-l_1\\ l_2\end{pmatrix} \\ & a^{l_1}b^{l_2}c^{\hat{L}-(l_1+l_2)}\frac{x^I}{I!}\frac{y^J}{J!}\frac{z^K}{K!}\frac{t^{2\hat{L}+N_4}}{(2\hat{L}+N_4)!} \\ & u_{I+2l_1,J+2l_2,K+2(\hat{L}-l_1-l_2),N_4}\,. \end{aligned} \tag{51}$$

Using the correspondences

$$I = 2\hat{I} + N_1 \qquad \hat{I} = i - l_1$$

$$J = 2\hat{J} + N_2 \qquad \hat{J} = j - l_2 \tag{52}$$

$$K = 2\hat{K} + N_3 \qquad \hat{K} = k - l_3$$

$$\hat{L} = l_1 + l_2 + l_3$$

we rearrange the series in (51) to obtain

$$u = \sum_{N_i=0}^{1} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \psi_{2i+N_1,2j+N_2,2k+N_3,N_4} u_{2i+N_1,2j+N_2,2k+N_3,N_4} \tag{53}$$

where

$$\psi_{2i+N_1,2j+N_2,2k+N_3,N_4} = \sum_{l_1=0}^{i} \sum_{l_2=0}^{j} \sum_{l_3=0}^{k} \begin{pmatrix} l_1+l_2+l_3 \\ l_1 \end{pmatrix} \begin{pmatrix} l_2+l_3 \\ l_2 \end{pmatrix}$$

$$a^{l_1} b^{l_2} c^{l_3} \frac{x^{2(i-l_1)+N_1}}{[2(i-l_1)+N_1]!} \frac{y^{2(j-l_1)+N_2}}{[2(j-l_2)+N_2]!} \frac{z^{2(k-l_3)+N_3}}{[2(k-l_3)+N_3]!}$$

$$\frac{t^{2(l_1+l_2+l_3)+N_4}}{[2(l_1+l_2+l_3)+N_4]!} . \tag{54}$$

We note that the basis solutions $\psi_{2i+N_1,2j+N_2,2k+N_3,N_4}$ are finite polynomials in x, y, z and t. For example,

$$\psi_{222} = \tfrac{1}{8}[x^2y^2z^2 + ay^2z^2t^2 + bx^2z^2t^2 + cx^2y^2t^2]$$

$$+[abz^2 + bcx^2 + acy^2]\tfrac{t^4}{4!} + \tfrac{abct^6}{5!} . \tag{55}$$

THREE-DIMENSIONAL ANISOTROPIC HEAT EQUATION

As a final example, we consider the three-dimensional anisotropic heat equation which is important in heat conduction and diffusion problems. Under appropriate variable transformations, this equation can become a free-particle Schrödinger equation or a paraxial wave equation in three-dimensions. Starting with

$$u_t = au_{xx} + bu_{yy} + cu_{zz} \tag{56}$$

and defining

$$u_{I,J,K,L} = \frac{\partial^I}{\partial x^I}\frac{\partial^J}{\partial y^J}\frac{\partial^K}{\partial z^K}\frac{\partial^L}{\partial t^L}u \tag{57}$$

we find

$$u_{I,J,K,L} = \frac{\partial^I}{\partial x^I}\frac{\partial^J}{\partial y^J}\frac{\partial^K}{\partial z^K}\left(a\frac{\partial^2}{\partial x^2} + b\frac{\partial^2}{\partial y^2} + c\frac{\partial^2}{\partial z^2}\right)^L u \; . \tag{58}$$

Using a binomial expansion gives

$$\begin{aligned} u_{I,J,K,L} &= \frac{\partial^I}{\partial x^I}\frac{\partial^J}{\partial y^J}\frac{\partial^K}{\partial z^K}\sum_{l_i=0}^{L}\sum_{l_2=0}^{L-l_1}\begin{pmatrix} L \\ l_1 \end{pmatrix}\begin{pmatrix} L-l_1 \\ l_2 \end{pmatrix} a^{l_1}b^{l_2}c^{L-l_1-l_2}\frac{\partial^{2l_1}}{\partial x^{2l_1}}\frac{\partial^{2l_2}}{\partial y^{2l_2}}\frac{\partial^{2(L-l_1-l_2)}}{\partial z^{2(L-l_1-l_2)}}u \\ &= \sum_{l_1=0}^{L}\sum_{l_2=0}^{L-l_1}\begin{pmatrix} L \\ l_1 \end{pmatrix}\begin{pmatrix} L-l_1 \\ l_2 \end{pmatrix} a^{l_1}b^{l_2}c^{L-l_1-l_2}u_{I+2l_1,J+2l_2,K+2(L-l_1-l_2)} \; . \end{aligned} \tag{59}$$

Proceeding as in the previous example we find

$$u = \sum_{N_i=0}^{1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\psi_{2i+N_1,2j+N_2,2k+N_3,0}u_{2i+N_1,2j+N_2,2k+N_3,0} \tag{60}$$

where

$$\psi_{2i+N_1,2j+N_2,2k+N_3,0} = \sum_{l_1=0}^{i}\sum_{l_2=0}^{j}\sum_{l_3=0}^{k}\begin{pmatrix} l_1+l_2+l_3 \\ l_1 \end{pmatrix}\begin{pmatrix} l_2+l_3 \\ l_2 \end{pmatrix} a^{l_1}b^{l_2}c^{l_3} \frac{x^{2(i-l_1)+N_1}}{[2(i-l_1)+N_1]!}\frac{y^{2(j-l_2)+N_2}}{[2(j-l_2)+N_2]!}\frac{z^{2(k-l_3)+N_3}}{[2(k-l_3)+N_3]!}\frac{t^{l_1+l_2+l_3}}{[l_1+l_2+l_3]!} \tag{61}$$

For example the first few basis functions for the heat equation are

$$\begin{aligned} &\psi_{0,0,0} = 1 \quad &&; \psi_{1,0,0} = x \quad &&; \psi_{0,1,0} = y \quad &&; \psi_{0,0,1} = z \\ &\psi_{1,1,0} = xy \quad &&; \psi_{1,1,1} = xyz \quad &&; \psi_{0,1,1} = yz \quad &&; \psi_{1,0,1} = xz \\ &\psi_{2,0,0} = \tfrac{x^2}{2} + at \quad &&; \psi_{0,2,0} = \tfrac{y^2}{2} + bt \quad &&; \psi_{0,0,2} = \tfrac{z^2}{2} + ct && \\ &\psi_{2,2,0} = \tfrac{x^2y^2}{4} + \tfrac{ay^2t}{2} + \tfrac{bx^2t}{2} + \tfrac{2abt^2}{2} && && && \end{aligned} \tag{62}$$

The specific examples of basis functions for the anisotropic heat equation are of course simpler than their counterparts for the anisotropic wave equation due to the presence of a first derivative only in the time variable.

IV. HELMHOLTZ SYMMETRY OPERATORS

In this section, we examine the relationship between basis solutions and symmetry operators of the two-dimensional Helmholtz equation.

From (22) and (23), the basis solutions of the Helmholtz equation in (9) can be split into four types, i.e.,

$$H_{2n,0} = \sum_{k=0}^{n} \frac{x^{2(n-k)}}{[2(n-k)]!} g_{2k}(y) \tag{63}$$

$$H_{2n+1,0} = \sum_{k=0}^{n} \frac{x^{2(n-k)+1}}{[2(n-k)+1]!} g_{2k}(y) \tag{64}$$

$$H_{2n,1} = \sum_{k=0}^{n} \frac{x^{2(n-k)}}{[2(n-k)]!} g_{2k+1}(y) \tag{65}$$

$$H_{2n+1,1} = \sum_{k=0}^{n} \frac{x^{2(n-k)+1}}{[2(n-k)+1]!} g_{2k+1}(y) \tag{66}$$

and we can write the solution of (9) in the form

$$u(x,y) = \sum_{n=0}^{\infty} u_{2n,0}H_{2n,0} + u_{2n+1,0}H_{2n+1,0} + u_{2n,1}H_{2n,1} + u_{2n+1,1}H_{2n+1,1}. \tag{67}$$

We know from [12] that the symmetry operators associated with the two-dimensional Helmholtz equation are

$$E, \;\; P_1 = \frac{\partial}{\partial x}, \;\; P_2 = \frac{\partial}{\partial y}, \;\; M = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \tag{68}$$

where E is the identity operator. Repeated application of these operators on a given solution of (9) will produce further solutions. We wish to point out the following relationship between these operators and the basis functions $H_{n,j}$.

Lemma 4:

$$\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right) H_{m-1,1} = e_m H_{m,0} + \sum_{i=1}^{m} a_i H_{m-i,0} + \sum_{i=1}^{m} b_i H_{m-i,1} \tag{69}$$

where $e_m = -m$, $a_i = a_i(m)$, and $b_i = b_i(m)$ are constants.

Proof: First note that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right) H_{m-1,1} = -\lambda^2 \left[y\frac{\partial H_{m-1,1}}{\partial x} - x\frac{\partial H_{m-1,1}}{\partial y}\right] \tag{70}$$

since $\Delta H_{\ell,j} = -\lambda^2 H_{\ell,j}$ for all $0 \le \ell$ and $0 \le j \le 1$. We wish to show that

$$\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right) H_{m-1,1} = e_m H_{m,0} + F \tag{71}$$

where F is the sum of terms of the form $x^l y^k g_r$ with $0 \le k \le 1$ and $l \le m-1$. We know that both $H_{m,0}$ and $\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right) H_{m-1,1}$ satisfy the Helmholtz equation. Once (71) is established we can use

$$\Delta F + \lambda^2 F = 0 \tag{72}$$

and Theorem 1 to get

$$F = \sum_{i=0}^{m} a_i H_{m-i,0} + b_i H_{m-i,1} \tag{73}$$

because the x derivatives of F of order m and higher are zero. Hence the proof will be complete when we show that (71) is true.

Case I. Assume that $m = 2n+1$ is odd. Then

$$H_{m,0} = H_{2n+1,0} = \sum_{k=0}^{n} \frac{x^{2(n-k)+1}}{[2(n-k)+1]!}\, g_{2k}(y) \tag{74}$$

and

$$H_{m-1,1} = H_{2n,1} = \sum_{k=0}^{n} \frac{x^{2(n-k)}}{[2(n-k)]!}\, g_{2k+1}(y)\ . \tag{75}$$

Using the fact that $\frac{\partial}{\partial y}\, g_{2k+1} = g_{2k};\ k \ge 0$, we can write

$$\begin{aligned} \left(y\tfrac{\partial}{\partial x} - x\tfrac{\partial}{\partial y}\right) H_{m-1,1} &= \left(y\tfrac{\partial}{\partial x} - x\tfrac{\partial}{\partial y}\right) \left[\textstyle\sum_{k=0}^{n} \frac{x^{2(n-k)}}{[2(n-b)]!}\, g_{2k+1}(y)\right] \\ &= y \textstyle\sum_{k=0}^{n} \frac{x^{2(n-k)-1}}{[2(n-k)]!}\, g_{2k+1}(y) - \sum_{k=0}^{n} \frac{x^{2(n-k)+1}}{[2(n-k)]!}\, g_{2k}(y)\ . \end{aligned} \tag{76}$$

But if

$$S(x,y) = \sum_{k=0}^{n} \frac{x^{2(n-k)+1}}{[2(n-k)]!}\, g_{2k}(y) = \sum_{k=0}^{n} \frac{2(n-k)+1\ x^{2(n-k)+1}}{[2(n-k)+1]!}\, g_{2k}(y), \tag{77}$$

then

$$S(x,y) = (2n+1)\ \sum_{k=0}^{n} \frac{x^{2(n-k)+1}}{[2(n-k)+1]!}\, g_{2k}(y) - \sum_{k=1}^{n} \frac{2k\ x^{2(n-k)+1}}{[2(n-k)+1]!}\, g_{2k}(y) \tag{78}$$

or

$$S(x,y) = (2n+1)\ H_{2n+1,0} - \sum_{k=1}^{n} \frac{2k\ x^{2(n-k)+1}}{[2(n-k)+1]!}\, g_{2k}(y)\ . \tag{79}$$

Combining (76) and (79), for $m = 2n+1$,

$$\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right) H_{m-1,1} + m H_{m,0} = F \tag{80}$$

where

$$\frac{\partial F}{\partial x^j} = 0 \quad \text{for} \quad j \geq m, \tag{81}$$

thus establishing (71).

Case II. Assume that $m = 2n$ is even. Again by Theorem 1,

$$H_{m,0} = H_{2n,0} = \sum_{k=0}^{n} \frac{x^{2(n-k)}}{[2(n-k)]!}\, g_{2k}(y) \tag{82}$$

and

$$H_{m-1,1} = H_{2n-1,1} = H_{2(n-1)+1,1} = \sum_{k=0}^{n-1} \frac{x^{2(n-k)-1}}{[2(n-k)-1]!}\, g_{2k+1}(y) \tag{83}$$

As in Case I, we find that

$$\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right) H_{m-1,1} = y \sum_{k=0}^{n-1} \frac{x^{2(n-k)-2}}{[2(n-k)-1]!}\, g_{2k+1}(y) - \sum_{k=0}^{n-1} \frac{x^{2(n-k)}}{[2(n-k)-1]!}\, g_{2k}(y). \tag{84}$$

But if

$$T(x,y) = \sum_{k=0}^{n-1} \frac{x^{2(n-k)}}{[2(n-k)-1]!}\, g_{2k}(y) = \sum_{k=0}^{n-1} \frac{2(n-k)x^{2(n-k)}}{[2(n-k)]!}\, g_{2k}(y), \tag{85}$$

then

$$T(x,y) = 2n \sum_{k=0}^{n-1} \frac{x^{2(n-k)}}{[2(n-k)]!}\, g_{2k}(y) + 2ng_{2n}(y) - \sum_{k=1}^{n-1} \frac{kx^{2(n-k)}}{[2(n-k)]!}\, g_{2k}(y) - 2ng_{2n}(y) \tag{86}$$

or

$$T(x,y) = 2nH_{2n,0} - 2ng_{2n}(y) - \sum_{k=1}^{n-1} \frac{kx^{2(n-k)}}{[2(n-k)]!}\, g_{2k}(y). \tag{87}$$

Combining (84) and (87) with $m = 2n$ gives

$$\begin{aligned} \left(y\tfrac{\partial}{\partial x} - x\tfrac{\partial}{\partial y}\right) H_{m-1,1} + mH_{m,0} \quad &= \quad y \textstyle\sum_{k=0}^{n-1} \frac{x^{2(n-k)-2}}{[2(n-k)-1]!}\, g_{2k+1}(y) \\ &\quad + 2ng_{2n}(y) + \textstyle\sum_{k=1}^{n-1} \frac{kx^{2(n-k)}}{[2(n-k)]!}\, g_{2k}(y) = F \end{aligned} \tag{88}$$

where F satisfies (72) and (81). Thus

$$F = \sum_{i=0}^{m-1} a_i H_{i,0} + b_i H_{i,1}\ . \tag{89}$$

∎

Theorem 5: Let $\phi_0 = \cos \lambda y$ and let $E = I,\ P_1 = \frac{\partial}{\partial x},\ P_2 = \frac{\partial}{\partial y},\ M = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$. Let S be the set of all functions of the form $\prod_{v=1}^{\infty} E^{I_v} P_1^{J_v} P_2^{K_v} M^{L_v} \phi_0;\ I_v, J_v, K_v, L_v = 0$ or 1. Then S is a complete set.

Proof: We show that the operators E, P_1, P_2, M can be used to generate the basis functions $H_{m,i}$ for $0 \leq m,\ 0 \leq i \leq 1$. Since these basis functions are complete, S must be complete also.

Beginning with the lower order $H_{m,i}$, recall that

$$\begin{aligned}
H_{0,0} &= \cos \lambda y = E\phi_0 \\
H_{0,1} &= \frac{\sin \lambda y}{\lambda} = -\frac{1}{\lambda^2} P_2 \phi_0 \\
H_{1,0} &= x \cos \lambda y = M\phi_0 \\
H_{1,1} &= \frac{x \sin \lambda y}{\lambda} = -\frac{1}{\lambda^2} P_2 M \phi_0
\end{aligned} \tag{90}$$

As an induction hypothesis, suppose that we can generate $H_{0,0}, H_{1,0}, \ldots, H_{m-1,0}$ and $H_{0,1}, \ldots, H_{m-1,1}$ from ϕ_0. Then by the previous lemma,

$$H_{m,0} = -\frac{1}{m}\left(MH_{m-1,1} - \sum_{i=1}^{m} a_i H_{m-i,0} + b_i H_{m-i,1} \right) \tag{91}$$

i.e., $H_{m,0}$ can be generated from ϕ_0. Furthermore

$$-\lambda^2 H_{m,1} = H_{m,1,xx} + H_{m,1,yy}. \tag{92}$$

But

$$H_{m,1,yy} = H_{m,0,y} = P_2 H_{m,0} \tag{93}$$

and

$$H_{m,1,xx} = H_{m-2,1}, \tag{94}$$

so

$$H_{m,1} = -\frac{1}{\lambda^2}\left[H_{m-2,1} + P_2 H_{m,0} \right], \tag{95}$$

i.e., $H_{m,1}$ can also be generated from ϕ_0 using E, P_1, P_2, and M. ∎

V. CONCLUSION

A general procedure for generating a complete basis of solutions for any linear constant coefficient PDE has been given. For many problems, especially for noneigenvalue PDEs, these 'Cauchy-Kowalewski' basis functions are finite polynomials and as such readily lend themselves to finite element and collocation type numerical approximations. For eigenvalue PDEs, however, the basis functions are infinite series which potentially limits their usefulness. This drawback presents only minor difficulties in the particular case of the the Helmholtz equation; by using spherical Bessel functions, simple recursions have been derived for the Helmholtz basis functions thus leading to easy procedures for their evaluation (see [10] for a discussion of this point). In general, the basis solutions obtained by the Cauchy-Kowalewski procedure are nonseparable in nature and quite useful in obtaining closed-form approximations of eigenfunctions and associated eigenvalues for domains with complex geometries.

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