Mathematics Notes

Note 93

17 March 2001

# Relationships Between Time- and Frequency-Domain Norms of Scalar Functions

Carl E. Baum
Air Force Research Laboratory
Directed Energy Directorate

Abstract

This paper addresses the relationship between time-domain waveforms and their Fourier transforms in terms of various norms of the two, specifically the 1-, 2-, and ∞-norms. As one might expect, norms in time domain can be used to provide some bounds concerning the norms in frequency domain, and conversely.

This work was sponsored in part by the Air Force Office of Scientific Research, and in part by the Air Force Research Laboratory, Directed Energy Directorate.

## 1. Introduction

Norms have been introduced to reduce the number of parameters characterizing transient waveforms to a few nonnegative numbers, which can be used in some kind of bounding sense [1-5]. This applies to both excitation and response waveforms and the transfer operators that relate them. Such transfer operators model the properties of electromagnetic systems. The present paper is concerned only with waveforms.

As it is common to consider the Laplace/Fourier transform of waveforms for their (complex-) frequency spectra, one can also consider such norms of these frequency-domain functions. This raises the question of the relation between the various norms in time and frequency domains, the subject of this paper.

2. Functional Norms

Consider now some generally complex scalar function $h(x)$ where $x$ is a real variable. The norm is an operator $\| \quad \|$ with the following properties [4]

$$\|h(x)\| \begin{cases} = 0 \ \mathit{iff} \ h(x) \equiv 0 \text{ or has zero "measure" per the particular norm} \\ > 0 \text{ otherwise} \end{cases}$$

$$\|\alpha\, h(x)\| = |\alpha| \, \|h(x)\| \quad , \quad \alpha \equiv \text{scalar} \tag{2.1}$$

$$\| h_1(x) + h_2(x) \| \leq \|h_1(x)\| + \|h_2(x)\|$$

Note that unlike N-component vectors, functions (which can be thought of as $\infty$-component vectors) need certain continuity requirements to avoid isolated points which contribute nothing to integrals.

A commonly used norm is the p-norm defined by

$$\|h(x)\|_p \equiv \left[ \int_{-\infty}^{\infty} |h(x)|^p \, dx \right]^{\frac{1}{p}} \quad , \quad 1 \leq p < \infty \tag{2.2}$$

with the special case of $p = \infty$ given by

$$\|h(x)\|_p = \sup_x |h(x)| \tag{2.3}$$

For present purposes we are primarily interested in $p = 1, 2, \infty$.

In [3] we discuss the concept of a "natural" norm based on the physical properties of systems of interest. For time-invariant systems we have the concept of a *time-invariant* norm, which takes the form

$$\|h(x - x_0)\| = \|h(x)\| \quad , \quad x_0 \text{ real} \tag{2.4}$$

where x in this case takes on the role of time. The $p$-norms all have this property.

Another interesting property of the $p$-norm is its dilation property

$$\left\| h(\chi x) \right\|_{p,x} = \left[ \int_{-\infty}^{\infty} \left| h(\chi x) \right|^p dx \right]^{\frac{1}{p}} = \left[ \chi^{-1} \int_{-\infty}^{\infty} \left| h(\chi x) \right|^p d(\chi x) \right]^{\frac{1}{p}} = \chi^{-\frac{1}{p}} \left\| h(x) \right\|_{p,x} \qquad (2.5)$$

$$\chi > 0$$

Note the inclusion of a second subscript to be clear that $x$, and not $\chi x$ is the integration variable. Note at this point that $\chi$ can also be negative with the result $|\chi|^{-\frac{1}{p}}$. With $x$ taking the role of time this exhibits the *time-reversal* invariance of the $p$-norm.

3, Hölder Inequality

We shall also make use of the Hölder inequality [4], which for functions takes the form

$$\left| \int_a^b h_1(x) h_2(x) dx \right| \le \left[ \int_a^b |h_1(x)|^{p_1} dx \right]^{\frac{1}{p_1}} \left[ \int_a^b |h_2(x)|^{p_2} dx \right]^{\frac{1}{p_2}} \tag{3.1}$$

$$1 = p_1^{-1} + p_2^{-1} \quad , \quad p_1 \ge 1 \quad , \quad p_2 \ge 1$$

Extending the integration from -∞ to +∞, and setting both functions as the same gives

$$\|h(x)\|_2^2 \le \|h(x)\|_{p_1} \, \|h(x)\|_{p_2} \tag{3.2}$$

which has the special case

$$\|h(x)\|_2^2 \le \|h(x)\|_1 \, \|h(x)\|_\infty \tag{3.3}$$

4. Frequency and Time

Now consider some time-domain function $f(t)$. In complex-frequency domain we have

$$\tilde{f}(s) \equiv \int_{-\infty}^{\infty} f(t)e^{-st}\,dt$$

$\sim \equiv$ two-sided Laplace transform over time t

$s \equiv \Omega + j\omega \equiv$ Laplace-transform variable or complex frequency

$$f(t) = \frac{1}{2\pi j}\int_{Br} \tilde{f}(s)e^{st}\,ds \text{ (inverse Laplace transform)} \tag{4.1}$$

$Br \equiv$ Bromwich contour parallel to $j\omega$ axis in $s$ plane in strip of convergence of two-sided Laplace transform

Setting $\Omega = 0$ (with $\omega$ real) defines the Fourier transform

$$\tilde{f}(j\omega) \equiv \int_{-\infty}^{\infty} \tilde{f}(j\omega)e^{-j\omega t}\,dt$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(j\omega)e^{j\omega t}\,d\omega \tag{4.2}$$

Note that the above assumes that $\tilde{f}(s)$ has no singularities on or to the right of the $j\omega$ axis, otherwise the contour of integration has to be appropriately modified.

Usually one is interested in real-valued time functions, which imply

$$\tilde{f}^*(s) = \tilde{f}(s^*)$$

$* \equiv$ complex conjugate

$$\tilde{f}^*(j\omega) = \tilde{f}(-j\omega) \tag{4.3}$$

$$f(t) = \frac{1}{\pi}\int_0^{\infty} \mathrm{Re}\left(\tilde{f}(j\omega)e^{j\omega t}\right)d\omega$$

$$= \frac{1}{\pi}\int_0^{\infty} \mathrm{Re}\left(\tilde{f}(-j\omega)e^{-j\omega t}\right)d\omega$$

Furthermore, our functions usually (but not always) start after some time $t_0$ and decay exponentially at late time insuring that all singularities are in the left half $s$ plane with the right half plane analytic in $s$.

In Section 2 we found the time-invariant result for p norms

$$\|f(t-t_0)\|_{p,t} = \|f(t)\|_{p,t} \quad , \quad t_0 \text{ real} \tag{4.4}$$

Going to frequency domain we have

$$\|\tilde{f}(j\omega)\|_{p,\omega} \equiv \left[\int_{-\infty}^{\infty} |\tilde{f}(j\omega)|^p \, d\omega\right]^{\frac{1}{p}} \tag{4.5}$$

From the shifting theorem of the Laplace transform we have

$$\int_{-\infty}^{\infty} f(t-t_0)e^{-st}dt = \tilde{f}(s)e^{-st_0} \tag{4.6}$$

Applying this in terms of the $p$-norm over $\omega$ we have

$$\begin{aligned} \left\|\tilde{f}(j\omega)e^{-j\omega t_0}\right\|_{p,\omega} &= \left[\int_{-\infty}^{\infty} \left|\tilde{f}(j\omega)e^{-j\omega t_0}\right|^p d\omega\right]^{\frac{1}{p}} \\ &= \left[\int_{-\infty}^{\infty} |\tilde{f}(j\omega)|^p \, d\omega\right]^{\frac{1}{p}} \\ &= \|\tilde{f}(j\omega)\|_{p,\omega} \end{aligned} \tag{4.7}$$

as the consequence of a time-invariant norm in frequency domain.

The dilation property (2.5) in time domain also carries over to frequency domain as

$$\int_{-\infty}^{\infty} f(\chi t)e^{-st}dt = \chi^{-1}\int_{-\infty}^{\infty} f(\chi t)e^{-\frac{s}{\chi}\chi t}d(\chi t)$$

$$= \chi^{-1}\tilde{f}\left(\frac{s}{\chi}\right)$$

$$\left\|\tilde{f}\left(\frac{j\omega}{\chi}\right)\right\|_{p,\omega} = \left[\int_{-\infty}^{\infty}\left|\tilde{f}\left(\frac{j\omega}{\chi}\right)\right|^{p} d\omega\right]^{\frac{1}{p}} \tag{4.8}$$

$$= \left[\chi\int_{-\infty}^{\infty}\left|\tilde{f}\left(\frac{j\omega}{\chi}\right)\right|^{p} d\left(\frac{\omega}{\chi}\right)\right]^{\frac{1}{p}}$$

$$= \chi^{\frac{1}{p}}\left\|\tilde{f}(j\omega)\right\|_{p,\omega}$$

Negative $\chi$ is also handled via $|\chi|^{\frac{1}{p}}$ and conjugate symmetry in (4.3).

## 5. 2-Norm for Time and Frequency

We have the well-known Parseval theorem [4] relating the 2-norms in frequency and time as

$$\left\| f(t) \right\|_{2,t} = \frac{1}{\sqrt{2\pi}} \left\| \tilde{f}(j\omega) \right\|_{2,\omega} \tag{5.1}$$

This can be considered in some contexts as equating energy in time and frequency.

The Hölder inequality (Section 3) then gives

$$\begin{aligned} \left\| f(t) \right\|_{2,t}^2 &= \frac{1}{2\pi} \left\| \tilde{f}(j\omega) \right\|_{2,\omega}^2 \le \left\| f(t) \right\|_{1,t} \left\| f(t) \right\|_{\infty,t} \\ \left\| \tilde{f}(j\omega) \right\|_{2,\omega}^2 &= 2\pi \left\| f(t) \right\|_{2,t}^2 \le \left\| \tilde{f}(j\omega) \right\|_{1,\omega} \left\| \tilde{f}(j\omega) \right\|_{\infty,\omega} \end{aligned} \tag{5.2}$$

The 2-norm relation between frequency and time then brings on some relations involving 1- and ∞-norms, mixing time and frequency norms.

## 6. Bounds for ∞-Norm in Time and 1-Norm in Frequency

Consider a bound as

$$\begin{aligned}
\|f(t)\|_{\infty,t} &= \sup_t |f(t)| = \sup_t \left| \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(j\omega) e^{j\omega t} d\omega \right| \\
&\le \sup_t \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \tilde{f}(j\omega) e^{j\omega t} \right| d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \tilde{f}(j\omega) \right| d\omega \\
&= \frac{1}{2\pi} \left\| \tilde{f}(j\omega) \right\|_{1,\omega}
\end{aligned} \tag{6.1}$$

This is useful provided the 1-norm in frequency exists. This requires that $\tilde{f}(j\omega)$ full off faster than $\omega^{-1}$ as $\omega \to \infty$ and that any singularities on the $j\omega$ axis be integrable (or just that $f(j\omega)$ be bounded). In (6.1) one can also avoid the use of supremum by choosing $t$ such that equality holds (assuming that the supremum is a maximum).

For a lower bound we have

$$\begin{aligned}
\|f(t)\|_{\infty,t} &= \sup_t |f(t)| = \sup_t \left| \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(j\omega) e^{j\omega t} d\omega \right| \\
&\ge \frac{1}{2\pi} \left| \int_{-\infty}^{\infty} \tilde{f}(j\omega) e^{j\omega t} d\omega \right| = |f(t)|
\end{aligned} \tag{6.2}$$

So choosing some arbitrary $t$ gives a lower bound. As a special case we have (setting $t = 0$)

$$\begin{aligned}
\|f(t)\|_{\infty,t} &\ge \frac{1}{2\pi} \left| \int_{-\infty}^{\infty} \tilde{f}(j\omega) d\omega \right| \\
&= \frac{1}{\pi} \left| \int_{0}^{\infty} \mathrm{Re}\left(\tilde{f}(j\omega)\right) d\omega \right|
\end{aligned} \tag{6.3}$$

for real $f(t)$ using (4.3). More generally we have

$$\|f(t)\|_{\infty,t} \geq \frac{1}{\pi}\left|\int_0^\infty \mathrm{Re}\left(\tilde{f}(j\omega)\right)e^{j\omega t}d\omega\right| \quad \text{(arbitrary real } t\text{)} \tag{6.4}$$

Summarizing we have

$$|f(t)| = \frac{1}{\pi}\left|\left|\int_0^\infty \mathrm{Re}\left(\tilde{f}(j\omega)e^{j\omega t}\right)d\omega\right|\right|_{\text{any real } t} \tag{6.5}$$

$$\leq \|f(t)\|_{\infty,t} \leq \frac{1}{2\pi}\left\|\tilde{f}(j\omega)\right\|_{1,\omega}$$

7. Bounds for 1-Norm in Time and ∞-Norm in Frequency

Consider a bound as

$$\begin{aligned}\left\|\tilde{f}(j\omega)\right\|_{\infty,\omega} &= \sup_{\omega}\left|\tilde{f}(j\omega)\right| = \sup_{\omega}\left|\int_{-\infty}^{\infty} f(t)e^{-j\omega t}d\omega\right| \\ &\leq \sup_{\omega}\int_{-\infty}^{\infty}\left|f(t)e^{-j\omega t}\right|dt = \int_{-\infty}^{\infty}\left|f(t)\right|dt \\ &= \left\|f(t)\right\|_{1,t}\end{aligned} \tag{7.1}$$

Except for the factor of $2\pi$ (from the Fourier-transform convention) then (7.1) is dual to (6.1).

Note that $\|f(t)\|_{1,t}$ also bounds $\left|\tilde{f}(0)\right|$ (the low-frequency content of the waveform), but this is a looser bound than the above. For the special case that $f(t)$ is unipolar (nonnegative or nonpositive), then we have

$$\begin{aligned}\left\|f(t)\right\|_{1,t} &= \int_{-\infty}^{\infty}\left|f(t)\right|dt = \left|\int_{-\infty}^{\infty} f(t)dt\right| \\ &= \left|\tilde{f}(0)\right| = \left\|f(j\omega)\right\|_{\infty,\omega} \\ f(t) &\equiv \text{unipolar waveform}\end{aligned} \tag{7.2}$$

giving an equality rather than an inequality.

For a lower bound we have

$$\begin{aligned}\left\|\tilde{f}(j\omega)\right\|_{\infty,\omega} &= \sup_{\omega}\left|\tilde{f}(j\omega)\right| = \sup_{\omega}\left|\int_{-\infty}^{\infty} f(t)e^{j\omega t}dt\right| \\ &\geq \left|\int_{-\infty}^{\infty} f(t)e^{j\omega t}dt\right| = \left|\tilde{f}(j\omega)\right|\end{aligned} \tag{7.3}$$

So choosing some arbitrary $\omega$ gives a lower bound. As a special case set $\omega = 0$ giving

$$\left\|\tilde{f}(j\omega)\right\|_{\infty,\omega} \geq \left|\int_{-\infty}^{\infty} f(t)\,dt\right| = \left|\tilde{f}(0)\right| \tag{7.4}$$

More generally we have for real $f(t)$

$$\left\|\tilde{f}(j\omega)\right\|_{\infty,\omega} \geq 2\left|\int_{0}^{\infty} f(t)\cos(\omega t)\,dt\right| \quad \text{(arbitrary real } \omega\text{)} \tag{7.5}$$

Summarizing we have

$$\begin{aligned} \left|\tilde{f}(j\omega)\right| &= 2\left|\left|\int_{0}^{\infty} f(t)\cos(\omega t)\,d\omega\right|\right|_{\text{any real } \omega} \\ &\leq \left\|\tilde{f}(j\omega)\right\|_{\infty,\omega} \leq \left\|f(t)\right\|_{1,t} \end{aligned} \tag{7.6}$$

8. Application of the Hölder Inequality

From (3.3) we have

$$\|f(x)\|_{2,t}^2 = \frac{1}{2\pi}\|\tilde{f}(j\omega)\|_{2,\omega}^2 \le \|f(t)\|_{1,t}\,\|f(t)\|_{\infty,t} \tag{8.1}$$

which bounds the 2-norm in frequency in terms of the 1- and ∞-norms in time. From (6.5) we have

$$\frac{1}{2\pi}\|\tilde{f}(j\omega)\|_{2,\omega}^2 \le \|f(t)\|_{1,t}\,\frac{1}{2\pi}\|\tilde{f}(j\omega)\|_{1,\omega} \tag{8.2}$$

giving

$$\|f(t)\|_{1,t} \ge \frac{\|\tilde{f}(j\omega)\|_{2,\omega}^2}{\|\tilde{f}(j\omega)\|_{1,\omega}} \tag{8.3}$$

as another lower bound. This involves the 1- and 2-norms over frequency. However, it is not as tight as the ∞-norm over frequency in (7.6) since from (3.3) we have

$$\|\tilde{f}(j\omega)\|_{\infty,\omega} \ge \frac{\|\tilde{f}(j\omega)\|_{2,\omega}^2}{\|\tilde{f}(j\omega)\|_{1,\omega}} \tag{8.4}$$

Summarizing we have

$$\|f(t)\|_{1,t} \ge \|\tilde{f}(j\omega)\|_{\infty,\omega} \ge \frac{\|\tilde{f}(j\omega)\|_{2,\omega}^2}{\|\tilde{f}(j\omega)\|_{1,\omega}} \tag{8.5}$$

Similarly from (3.3) we have

$$\|\tilde{f}(j\omega)\|_{2,\omega}^2 = 2\pi\|f(t)\|_{2,t}^2 \le \|\tilde{f}(j\omega)\|_{1,\omega}\,\|\tilde{f}(j\omega)\|_{\infty,\omega} \tag{8.6}$$

which bounds the 2-norm in time in terms of the 1- and ∞-norms in frequency. From (7.6) we have

$$2\pi \left\| f(t) \right\|_{2,t}^{2} \le \left\| \tilde{f}(j\omega) \right\|_{1,\omega} \left\| f(t) \right\|_{1,t} \tag{8.7}$$

giving

$$\left\| \tilde{f}(j\omega) \right\|_{1,\omega} \ge 2\pi \frac{\left\| f(t) \right\|_{2,t}^{2}}{\left\| f(t) \right\|_{1,t}} \tag{8.8}$$

as another lower bound involving the 1- and 2-norms over time. However, it is not as tight as the $\infty$-norm over time in (6.5) since from (3.3) we have

$$\left\| f(t) \right\|_{\infty,t} \ge \frac{\left\| f(t) \right\|_{2,t}^{2}}{\left\| f(t) \right\|_{1,t}} \tag{8.9}$$

Summarizing we have

$$\left\| \tilde{f}(j\omega) \right\|_{1,\omega} \ge 2\pi \left\| f(t) \right\|_{\infty,t} \ge 2\pi \frac{\left\| f(t) \right\|_{2,t}^{2}}{\left\| f(t) \right\|_{1,t}} \tag{8.10}$$

## 9. Concluding Remarks

The results included here are quite general, and provide some simple bounds for norms one may not wish to calculate in terms of other norms one may have already obtained. In a previous paper [3] we have considered window norms for a time interval $t_1 \le t \le t_2$. Since this is equivalent to the norm of a waveform function $f(t)\left[u(t-t_1) - u(t-t_2)\right]$, then the present results apply to such functions as well. However, the Fourier transform of such a window function does not have the same form as a window in frequency domain. Of course the p-norm of such a windowed function is bounded by the p-norm of the (unwindowed) function.

References


1. C. E. Baum, "Transfer of Norms Through Black Boxes", Interaction Note 462, October 1987; Proc. Zurich EMC Symposium, March 1989, pp. 157-162.

2. C. E. Baum, "Comparative System Response to Resonant and Unipolar Waveforms", Interaction Note 509; Proc. Int'l. Sym. On Electromagnetic Compatibility, Magdeburg, Germany, October 1999, pp. 15-20.

3. C. E. Baum, "A Time-Domain View of Choice of Transient Excitation Waveforms for Enhanced Response of Electronic Systems", Interaction Note 560, September 2000.

4. C. E. Baum, "Norms of Time-Domain Functions and Convolution Operators", Mathematics Note 86, December 1985; ch. 2, pp. 31-55, in H. N. Kritikos and D. L. Jaggard (eds.), *Recent Advances in Electromagnetic Theory*, Springer-Verlag, 1990.

5. C. E. Baum, "Energy Norms and 2-Norms", Mathematics Note 89, April 1988; ch. 11.4, pp. 498-508, in H. Kikuchi (ed.), *Environmental and Space Electromagnetics*, Springer-Verlag, 1991.