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Correction of Time-Domain Data in Special Cases Where the Inverse Transfer Functions are Analytic Time-Domain Operators

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Abstract

This paper considers some techniques for correcting nonconstant transfer functions (corresponding to nondelta-function temporal operators) directly in time domain. Depending on the analytic form in the Laplace-transform (complex-frequency) domain, appropriate temporal deconvolution operators are found. Several examples are considered.

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1. Introduction

In processing of electromagnetic scattering data associated with an incident broadband pulse, there is the problem of removing the filtering of this data by the radar antenna(s), the characteristics of the incident pulse, and other electronic equipment (such as directional couplers). One approach to this problem is to construct, transfer functions (frequency domain) corresponding to these various factors. Then by use of numerical Fourier transforms one transforms the radar data, divides by the transfer functions and numerically inverse Fourier transforms to obtain the delta-function response of the target.

One of the advantages of looking at the delta-function response is the separation of the target response from that of other scatterers (clutter) by looking only at a time window corresponding to the time of return of the target signal. Such time separation is corrupted by the temporal convolution corresponding to the transfer functions mentioned above. There is also the inevitable problem of noise in the data as well. This can affect the accuracy of the deconvolution required by making the above transfer functions correspond as closely as practical to single delta functions in time (i.e., minimum dispersion).

An alternate approach, applicable in some cases, is to use analytic deconvolution directly in the time domain. This applies to the cases where the inverse of the transfer functions mentioned above can be written analytically in the time domain, and where the form that the deconvolution takes is relatively simple. This paper explores a few such cases.

2. The Typical Passive Integrator

As a simple case, let us consider a simple passive integrator (RC integrator) characterized by

$$
\begin{aligned}
\tilde{V}_{out}(s) &= [1+s\tau]^{-1}\tilde{V}_{in}(s) \\
\tau &\equiv \text{time constant} > 0 \\
\sim &\equiv \text{two-sided Laplace transform over time } t \\
s &= \Omega + j\omega \equiv \text{Laplace-transform variable or complex frequency}
\end{aligned}
\tag{2.1}
$$

What we wish to obtain is the time integral of $V_{in}(t)$. This is usually expressed as an approximation

$$
\begin{aligned}
\tilde{V}_{out}(t) &\simeq \frac{1}{\tau}\int_0^t V_{in}(t')dt' \ \text{ for } t \ll \tau \\
V_{in}(s) &= 0 \ \text{ for } t < 0
\end{aligned}
\tag{2.2}
$$

A more accurate approach observes that the inverse of the transfer function is a convolution operator

$$
\begin{aligned}
1 + s\tau \ &\leftrightarrow \ \left[\delta(t) + \tau\frac{d}{dt}\delta(t)\right]\circ \\
\circ &\equiv \text{convolution with respect to time}
\end{aligned}
\tag{2.3}
$$

Thus we have

$$
\begin{aligned}
\tilde{V}_{in}(s) &= [1+s\tau]\tilde{V}_{out}(s) \\
\int_0^t V_{in}(t')dt' &= [\tau\delta(t) + u(t)] \circ V_{out}(t) \\
&= \tau V_{out}(t) + \int_0^t V_{out}(t')dt'
\end{aligned}
\tag{2.4}
$$

If we have some data stream for $V_{out}(t)$, then this can be corrected by the addition of a simple integration to give a more accurate version of the time integral of $V_{in}(t)$.

3. Time-Domain Directional Coupler

A previous paper [3] considers a special type of time-domain directional coupler. See this paper for the development of a two-conductor (plus reference) ideal transmission line. Here we consider only the fully symmetric case for which we have

$$\left(Z_{c_{n,m}}\right) = \begin{pmatrix} Z_{c_{1,1}} & Z_{c_{1,2}} \\ Z_{c_{1,2}} & Z_{c_{1,1}} \end{pmatrix} \equiv \text{characteristic impedance matrix}$$

$$\left(X_{n,m}\right) = \frac{1}{R}\left(Z_{c_{n,m}}\right)$$

$R \equiv$ resistance loading all four ports (e.g., 50 Ω)

$$\det\left(\left(Z_{c_{n,m}}\right)\right) = Z_{c_{1,1}}^2 - Z_{c_{1,2}}^2 = R^2$$
$$\det\left(\left(X_{n,m}\right)\right) = X_{1,1}^2 - X_{1,2}^2 = 1 \tag{3.1}$$

The port convention has

port 1 ≡ port into which a pulse is launched

port 2 ≡ adjacent port at same end as port 1

port 3 ≡ port out of which the pulse is propagated to a load (e.g., radar antenna, well terminated)

port 4 ≡ port adjacent to port 3 which receives the reflected pulse to be recorded (e.g., a radar backscatter) and receives *no* signal from port 1 (3.2)

In [3] it is observed that for a time T, the round-trip transit time through the coupler, with

$$T = 2\frac{\ell}{\mathrm{v}}$$

$\ell \equiv$ coupler length

v = propagation speed ($\leq c$) in the lossless, dispersionless dielectric medium (3.3)

the pulse received in port 4 is a faithful replica of the pulse presented back to port 3. The same is true for the pulse from port 1 to port 3. In a radar application this leaves the antenna transfer function as an additional problem for possible deconvolution.

The scattering matrix elements of interest are

$$\tilde{S}_{3,1}(s) = \left[X_{1,1}\sinh\left(\frac{sT}{2}\right) + \cosh\left(\frac{sT}{2}\right)\right]^{-1}$$
$$\tilde{S}_{4,3}(s) = \tilde{S}_{3,1}(s)\, X_{1,2}\sinh\left(\frac{sT}{2}\right) \tag{3.4}$$
$$\tilde{S}_{4,1}(s) = 0 = \tilde{S}_{1,1}(s)$$

and the product of the first two of these. For convenience we define

$$\xi \equiv \frac{1 - X_{1,1}}{1 + X_{1,1}} < 0 \tag{3.5}$$

Rearranging we have

$$\tilde{S}_{3,1}(s) = 2\left[\left[1 + X_{1,1}\right]e^{\frac{sT}{2}} + \left[1 + X_{1,1}\right]e^{-\frac{sT}{2}}\right]^{-1}$$
$$= \frac{2e^{-\frac{sT}{2}}}{1 + X_{1,1}}\left[1 + \xi e^{-sT}\right]^{-1} \tag{3.6}$$

Expand the last part as a geometric series

$$\tilde{S}_5(s) \equiv \left[1 + \xi e^{-sT}\right]^{-1} = \sum_{n=0}^{\infty}[-\xi]^n e^{-nsT} \tag{3.7}$$

In time domain this is

$$\tilde{S}_5(t) = \sum_{n=0}^{\infty}[-\xi]^n \delta(t - nT) \tag{3.8}$$

So except for the delay of $T/2$ in (3.6) this is a successive set of delta functions, each $-\xi$ (positive) in decreasing amplitude times the previous and delayed by successive times T.

Note that if we delay S_5 by time T and multiply by ξ (negative) we have

$$\xi S_5(t-T) = -\sum_{n=0}^{\infty}[-\xi]^{n+1}\delta(t-[n+1]T) = -\sum_{n=1}^{\infty}[-\xi]^{n}\delta(t-nT) \tag{3.9}$$

We then find

$$\begin{aligned} S_5(t) + \xi S_5(t-T) &= \delta(t) \\ \tilde{S}_5(s)\left[1 + \xi e^{-sT}\right] &= 1 \end{aligned} \tag{3.10}$$

So taking a signal coming from the coupler and multiplying by ξ, delaying by T and adding, removes the effect of $S_{3,1}$ except for a constant multiplier and delay by $T/2$, as

$$S_{3,1}(t) + \xi S_{3,1}(t-T) = \frac{2}{1+X_{1,1}}\delta\left(t-\frac{T}{2}\right) \tag{3.11}$$

Continuing we have

$$\begin{aligned} \tilde{S}_{4,3}(s) &= \frac{X_{1,2}}{1 + X_{1,1}}\tilde{S}_6(s) \\ \tilde{S}_6(s) &= \left[1 + \xi e^{-sT}\right]^{-1}\left[1 - e^{-sT}\right] \end{aligned} \tag{3.12}$$

Apply the previous procedure (as in (3.10))

$$\tilde{S}_7(s) = \tilde{S}_6(s) + \xi e^{-sT}\tilde{S}_6(s) = 1 - e^{-sT} \tag{3.13}$$

Thus the previous shift by T, multiplication by ξ, and adding removes the denominator again. In time domain this leaves

$$\begin{aligned} S_7(t) &= \delta(t) - \delta(t-T) \\ S_{4,3}(t) + \xi S_{4,3}(t-T) &= \frac{X_{1,2}}{1 + X_{1,1}}S_7(t) \end{aligned} \tag{3.14}$$

To deconvolve the remaining $S_7(t)$ one can observe

$$S_8(t) = \delta(t) - \delta(t-Nt) = \sum_{n=0}^{N-1} S_7(t-nT) \tag{3.15}$$

By appropriate choice of N the second delta function can be moved into an appropriate time where there are ideally no clutter signals which could shift into the time window of interest for the target.

Noting that scalar convolution integrals commute, we can remove the effects of $S_{3,1}$ and $S_{4,3}$ by a succession of operations, in any order. There are two operations of multiply by ξ, shift by T, and add. Then there is the succession of shift and adds N - 1 times in (3.15). Ideally this leaves us with an operator

$$\frac{2X_{1,2}}{\left[1 + X_{1,1}\right]^2}\left[\delta\left(t-\frac{T}{2}\right) - \delta\left(t-\left[N + \frac{1}{2}\right]t\right)\right] \tag{3.16}$$

where the second delta function can be neglected by considerations of the received time-domain waveform. This leaves a signal multiplied by a constant factor and shifted by $T/2$.

Another interpretation of (3.16) concerns what might be called an "equivalent directional coupler". Comparing S_8 to S_7 the double transit time of T has been extended to NT, effectively making the coupler N times as long.

4. Ideal Reflector Impulse Radiating Antenna

A reflector impulse radiating antenna (IRA) has a transmitted waveform when driven by a step function pulse $V_0 u(t)$ as [4]

$$
\begin{aligned}
V_f(t) &= r\,E_f(t) = V_0 \frac{h_a}{2\pi c f_g} R(t) \quad \text{(far "voltage")} \\
R(t) &= \frac{1}{T}\left[-u(t) + u(t-T)\right] + \delta_a(t-T) \\
T &= \frac{2F}{c} \text{ (a round trip time on the antenna)} \\
F &= \text{focal length} \\
c &= \left[\mu_0 \varepsilon_0\right]^{-1/2} \equiv \text{speed of light}
\end{aligned}
\tag{4.1}
$$

Here only $R(t)$ concerns us. The various factors are explained in the references in [5]. The δ_a is an approximate delta function (a delta function in the limit of $r \to \infty$). $R(t)$ also characterizes such an antenna in reception (an approximate replicator). The step functions form what is called the prepulse. There is also a postpulse [2] (more complicated) which is not considered here.

The problem is to remove the step functions and leave a delta function remaining. Transforming we have

$$
\begin{aligned}
\tilde{R}(s) &= \frac{1}{sT}\left[-1 + e^{-sT}\right] + e^{-sT} \\
e^{-sT}\tilde{R}^{-1}(s) &= \left[\frac{1}{sT}\left[1 - e^{sT}\right] + 1\right]^{-1} \\
&= \frac{1}{1 + \frac{1}{sT}}\left[1 - \frac{1}{sT}\left[1 + \frac{1}{sT}\right]^{-1} e^{sT}\right]^{-1} \\
&= \frac{sT}{sT + 1}\sum_{n=0}^{\infty}\left[sT + 1\right]^{-n} e^{nsT}
\end{aligned}
\tag{4.2}
$$

This form has some difficulties. Besides the delay by T, in time domain the leading term is a derivative. Essentially this is keying on the initial step function instead of the later delta function.

An alternate form has

$$e^{-sT}\tilde{R}^{-1}(s) = \sum_{n=0}^{\infty}[sT]^{-n}\left[1-e^{sT}\right]^{n}$$
$$= 1 + \frac{1}{sT}[1 - sT] + \frac{1}{[sT]^2}[1 - sT]^2 + \cdots \tag{4.3}$$

In this form we have the leading delta function in time domain, but the successive correction terms are rather complicated.

Let us now approach this differently. We can characterize *R(t)* by a recursion beginning with

$$\tilde{V}_{out}(s) = e^{-sT}\tilde{V}_{in}(s) + \frac{1}{sT}\left[-1 + e^{-sT}\right]\tilde{V}_{in}(s)$$
$$V_{in}(t) \equiv \text{signal into “filter”}$$
$$V_{out}(t) \equiv \text{signal out of “filter”} \tag{4.4}$$

This is changed to

$$\tilde{V}_{in}(s) = e^{sT}\tilde{V}_{out}(s) + \frac{1}{sT}\left[e^{-sT} - 1\right]\tilde{V}_{in}(s)$$
$$\tilde{V}_{in}(t) = \tilde{V}_{out}(t+T) + \frac{1}{T}\left[\int_{-\infty}^{t}\tilde{V}_{in}(t'+T)dt' - \int_{-\infty}^{t}V_{in}(t')dt' - \right] \tag{4.5}$$
$$= V_{out}(t+T) + I_1(V_{in}(t))$$
$$I_1(V_{in}(t)) = \frac{1}{T}\int_{t}^{t+T}V_{in}(t')dt'$$

Define an integral operator

$$I_1(\cdots) \equiv \frac{1}{T}\int_{t}^{t+T}(\cdots)dt_i$$

$$I_1^n(\cdots) \equiv n = \text{fold multiple integral}$$
$$= \frac{1}{T^n}\int_{t}^{t+T}\int_{t_1}^{t_1+T}\cdots\int_{t_{n-1}}^{t_{n-1}+T}(\cdots)dt_n\cdots dt_1 \tag{4.6}$$

Applying this recursively we have

$$\begin{aligned} V_{in}(t) &= V_{out}(t+T) + I_1(V_{in}(t)) \\ &= V_{out}(t+T) + I_1(V_{out}(t+T)) + I_1(V_{in}(t)) \\ &= V_{out}(t+T) + \sum_{n=1}^{\infty} I_1^n\left(V_{out}(t+T)\right) \end{aligned} \tag{4.7}$$

So the inverse operator to $R(t)$ can be written in the form

$$\begin{aligned} V_{in}(t) &= A(V_{out}(t)) \\ A(\cdots) &= \delta(t+T) \circ + \sum_{n=1}^{\infty} I_2^n(\cdots) \\ I_2(\cdots) &= \frac{1}{T} \int_{t+T}^{t+2T} (\cdots)\,dt' \end{aligned} \tag{4.8}$$

Alternately we can deal in terms of $V_{out}(t+T)$ without the additional time shift.

This *formal* solution may have some difficulty in implementation due to the infinite series of repeated integrals. This may depend on whether or not a few terms are adequate for implementation in appropriate circumstances. This may also depend on the time duration for which it is to be applied.

5. Ideal Lens Impulse Radiating Antenna

A lens IRA is basically a TEM horn with a lens at the aperture to make the aperture fields a plane wave. The ideal transfer function given in [1, 4] is

$$
\begin{aligned}
V_f(t) &= r\,E_f(t) = -V_0 \frac{h}{4\pi c f_g} L(t) \\
L(t) &= \delta(t) + \frac{1}{T}\left[-u(t) + u(t-T)\right] \\
T &\equiv \text{ round trip transit time}
\end{aligned}
\tag{5.1}
$$

This is basically a time-reversal of $R(t)$.

Following the procedure in the previous section we have

$$
\begin{aligned}
V_{out}(t) &= V_{in}(t) + \frac{1}{T}\left[-\int_{-\infty}^{t} V_{in}(t')\,dt' + \int_{-\infty}^{t} V_{in}(t'-T)\,dt'\right] \\
&= V_{in}(t) - \frac{1}{T}\int_{t-T}^{t} V_{in}(t')\,dt' \\
V_{in}(t) &= V_{out}(t) + I_3\left(V_{in}(t)\right) \\
I_3(\cdots) &= \frac{1}{T}\int_{t-T}^{t} (\cdots)\,dt' \\
V_{in}(t) &= \delta(t) \circ V_{out}(t) + \sum_{n=1}^{\infty} I_3^n\left(V_{out}(t)\right) \\
\tilde{L}^{-1}(s) &\leftrightarrow \delta(t) \circ + \sum_{n=1}^{\infty} I_3^n(\cdots)
\end{aligned}
\tag{5.2}
$$

As one can see, this is quite similar to the previous result.

6. Concluding Remarks

Here we have analytic formulas for deconvolution in the time domain. These have varying degrees of simplicity. The shift-and-add form for the directional coupler is simpler than the recursive integral formulas for the IRAs.

Such analytic procedures will not solve all deconvolution problems. To the degree that they remove some of the dispersion in the data, the corrected data can perhaps be more accurately processed by numerical Fourier transforms since the remaining transfer function to be unfolded then corresponds more closely to a delta function.

References

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