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Symmetry in Electromagnetics

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Σύμμετροι αἱ
τῷ αὐτῷ μέτρῳ
μετρούμεναι.
Aristotle

ABSTRACT

Symmetry is a powerful concept which has found application in various areas such as quantum mechanics and crystallography. It has also found use in electromagnetics and this paper goes through such applications, particularly as might be used for antennas and scatterers . Such symmetries include those in the Maxwell equations (duality, reciprocity), those in geometric symmetry (rotation, reflection, translation, similarity) and combinations of these. In 4-dimensional space/time additional symmetries are also found. These symmetries are cast in a group theoretic form to help outline the various possibilities.

CONTENTS

I. INTRODUCTION

The concept of symmetries has been a rich source of inspiration that has been imbedded in our culture since antiquity. Although the aesthetics of the concepts of symmetry has been a source of continuous fascination both in the arts and in science , it was not until 1830 that the mathematical foundations were laid out by the French mathematician Galois who examined the symmetries of the solutions of polynomials of fifth degree and higher. Progress followed rapidly in the latter part of the early part of the the nineteenth century when it was realized that crystal shapes can be categorized through the elementary concepts of group theory. The fusion of the ideas in geometry and symmetry with physics has proven to be extraordinarily fruitful and has produced the modern foundations of Physics. H. Weyl [9.23] and A.V. Shubnikov and V.A. Koptsik [9.20] have beautifully written accounts of the early days of group theory .

Presently research in group theory finds itself in two classical entrenched fronts and a newcomer. The two classical areas of activity are the abstract mathematical world and the mathematical foundations of quantum mechanics . The newcomer is the electrical engineering community that is slowly beginning to realize that concepts of symmetry and group theory can provide powerful analytical tools . Applications were found early starting with the celebrated Babinet's principle where the concept of duality was found useful in drawing parallels between electric and magnetic sources. The principle of equivalence can also be thought as having its roots in the ideas of symmetry. The statement that a Huygens source is equivalent to a combined magnetic and electric current source is a statement of the symmetry of the Maxwell equations. The applications of the mathematical tools of symmetry which is group theory are slowly appearing in the electrical engineering literature. Montgomery and Dicke [4.14] examined multiport microwave networks using group theoretic concepts. The area were symmetry is beginning to play an important role is scattering. Group theoretic concepts are acquiring a significant role in the simplification of the scattering calculations by reducing the dimensionality of the resulting impedance matrices. B. Kibner and A. Kotlyar [1.10] and later D. Cohoon [1.3] have successfully used group theory to reduce the cost of solving scattering problems.C. Baum and his coworkers [1.1,1.2,2.1-3,2.5,2.8,4.1-8,6.1-2,9.2] have introduced group theoretic concepts for the characterization of scatterers and synthesis of electromagnetic devices. H. Kritikos [1.11,4.10,8.5] has introduced the point groups into antenna synthesis. N. Engeta [7.3] has shown that material chiral symmetry alone can sustain two distinct propagating modes.

While in physics and chemistry group theory is introduced to describe nature based on observations of symmetry in nature (e.g. molecules ,crystals, elementary particles,etc), in electrical engineering the motivation is different. Here we are not necessarily given symmetry as a fact ;we wish to construct symmetry into various devices so that desirable properties will result. A systematic investigation of symmetry in antennas ,scatterers , properties of various media as well as in combination with symmetries inherent in Maxwell's equations allows one to synthesize (design) new types of devices, the desirable properties not being otherwise obvious.

II. ELECTROMAGNETICS, SYMMETRY AND GROUPS

Our interest is that of exploiting various symmetries that are present (or can be imposed) in electromagnetic problems. Such symmetries take the form of invariance of the result to various transformations in the problem. Some of these invariances are found in the very form of the Maxwell equations.

$$\nabla \times \mathbf{E}(\mathbf{r},t) = -\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t) - \mathbf{J}_m(\mathbf{r},t)$$

$$\nabla \times \mathbf{H}(\mathbf{r},t) = \frac{\partial}{\partial t}\mathbf{D}(\mathbf{r},t) + \mathbf{J}(\mathbf{r},t) \tag{2.1}$$

Note the inclusion of the magnetic current density $\mathbf{J}_m$ for symmetry. This common artifice allows one to introduce the concept of of equivalent magnetic current and charges, not from the point of view of whether such things are physically present , but as a mathematical convenience. . For example , one may specify boundary conditions for the electric field in the form of a magnetic frill. Section III develops this kind of symmetry on electric/magnetic interchange which is called duality.

Besides the Maxwell equations one needs the constitutive relations which in a simple form are

$$\tilde{\mathbf{D}}(\mathbf{r},s) = s\tilde{\varepsilon}(s)\tilde{\mathbf{E}}(\mathbf{r},s) , \quad \tilde{\mathbf{J}}(\mathbf{r},s) = \tilde{\sigma}(s)\tilde{\mathbf{E}}(\mathbf{r},s) \tag{2.2}$$

$$\tilde{\mathbf{B}}(\mathbf{r},s) = s\tilde{\mu}(s)\tilde{\mathbf{H}}(\mathbf{r},s) , \quad \tilde{\mathbf{J}}_m(\mathbf{r},s) = \tilde{\sigma}_m(s)\tilde{\mathbf{H}}(\mathbf{r},s) \tag{2.3}$$

where the constitutive parameters are

$\tilde{\varepsilon}(s) \equiv$ permittivity, $\tilde{\sigma}(s) \equiv$ conductivity

$\tilde{\mu}(s) \equiv$ permeability , $\tilde{\sigma}_m(s) \equiv$ magnetic conductivity

which can be combined in the form $s\tilde{\varepsilon}(s) + \tilde{\sigma}(s)$ and $s\tilde{\mu}(s) + \tilde{\sigma}_m(s)$ if desired. The current densities in general include source terms as well. Note the introduction of the two sided Laplace Transform as

$$\tilde{f}(s) \equiv \int_{-\infty}^{\infty} f(t)e^{-st}dt \ , \ f(t) \equiv \frac{1}{2\pi j}\int_{Br} \tilde{f}(s)e^{st}ds \tag{2.4}$$

$s = \Omega + j\omega \ \equiv$ Laplace-Transform variable or complex frequency

$Br \equiv$ Bromwich contour in strip of convergence

This presupposes two symmetries : linearity (amplitude scaling symmetry) and time invariance (time translation symmetry). In time domain the constitutive parameters take the special form of convolution operators.

The constitutive parameters can be 3X3 matrices. If these matrices are symmetric (note symmetry again) , a special case of which is the scalar form in (2.2) , then we have the important electromagnetic symmetry known as reciprocity. This is discussed along with the energy theorems of similar form in section IV. Another interesting form of constitutive equations mixes the electric and magnetic terms giving a chiral medium which separates waves into right - and left -handed waves with different speeds. This form of symmetry breaking is discussed in section V and related to removal of reflection symmetry in the earlier duality discussion.

Next we introduce geometric symmetries (such as for antennas,scatterers and other electromagnetic objects) in terms of the point symmetry groups (rotation and reflection) in section VI. This leads a host of possible geometries of interest , the different types of symmetry being illustrated by a few examples involving scattering , measurements, capacitors and antenna arrays.

This sets the stage for combining the duality symmetry with point symmetry in the form of the Babinet's principle in section VII. Complementary planar structures lead to self-complementarity , self-inverse , and self-rotated-inverse structures with higher orders of symmetry. Special cases of self-complementary antennas and resistors are illustrated.

Continuing with the geometrical symmetries section VIII considers translation as well as rotation and reflection , giving the space groups . These are well known to be important for crystals , but also important for electromagnetics. Periodic structures such as periodically loaded waveguides are a common electromagnetic example. A more general kind of geometric symmetry involves similarity as discussed in section IX. In this case not only is there rotation but uniform dilation (change of scale) in the summetry operation. This is important for log-periodic and spiral structures which have been used for "frequency - independent " antennas.

Increasing the number of dimensions to four we have the Minkowski space in section X. This leads to the Lorenz and Poincare groups considering the symmetry between various inertial frames as well as translation, rotation and reflection. Various conservation laws under Lorenz transformation are discussed ,including the role of duality in this context.

An underlying theme for symmetry discussion is group theory . A <u>group</u> $\mathcal{G}$ is a set

$$\mathcal{G} = \left\{ (\mathcal{G})_\ell \right\} \ , \ \ell = 1,2,.. \tag{2.5}$$

The number of elements may be finite or infinite. If the number is finite , say ℓ_0 , this is the <u>order</u> of the group . A group is required to have the following properties

(a) For all $(\mathcal{G})_\ell$, $(\mathcal{G})_{\ell'}$

$(\mathcal{G})_\ell \, (\mathcal{G})_{\ell'} \in \mathcal{G}$ (Group operation)

(b) Associativity

$$(\mathcal{G})_\ell \, ((\mathcal{G})_{\ell'} \, (\mathcal{G})_{\ell''}) = ((\mathcal{G})_\ell \, (\mathcal{G})_{\ell'} \,)(\mathcal{G})_{\ell''} \tag{2.6}$$

(c) Identity element (1) (being one of the $(\mathcal{G})_\ell$)

$(1) \, (\mathcal{G})_\ell = (\mathcal{G})_\ell \cdot (1) = (\mathcal{G})_\ell$ for all ℓ

(d) Inverse

For each ℓ there is an ℓ' $\ni$

$(\mathcal{G})_\ell \, (\mathcal{G})_{\ell'} = (\mathcal{G})_{\ell'} \, (\mathcal{G})_\ell = (1)$

$(\mathcal{G})_{\ell'} = (\mathcal{G})_\ell^{-1}$ (inverse)

Here the group operation is often referred to as multiplication (indicated by placing a group element to the left of another group element) . However , this operation can correspond to various kinds of operation and group elements can be considered as operators.

The notation is suggestive in that one can have a <u>matrix representation</u> of a group , in which case one associates a square matrix (dxd) with each group element (d being the dimension of the representation) . The multiplication is the dot product sense (contraction) giving

$$(\mathcal{G})_\ell \rightarrow (\mathcal{G}^{(r)}_{n,m})_\ell$$

$$(\mathcal{G})_\ell \, (\mathcal{G})_{\ell'} \rightarrow (\mathcal{G}^{(r)}_{n,m})_\ell \bullet (\mathcal{G}^{(r)}_{n,m})_{\ell'} = (\mathcal{G}^{(r)}_{n,m})_{\ell''} \text{ for some } \ell'' \tag{2.7}$$

$$(1) \rightarrow \begin{pmatrix} 1_1 & & 0 \\ & \cdot & \\ 0 & & \cdot_1 \end{pmatrix} = (1_{n,m})$$

$$(\mathcal{G}^{(r)}_{n,m})^{-1}_{\ell} = (\mathcal{G}^{(r)}_{n,m})_{\ell''} \text{ for some } \ell''$$

Note the superscript (r) indicating the rth representation . By a similarity transformation a given representation can be changed to another one, so (except for r=1) the representation is not unique. Some representations are preferred over others . Specifically irreducible representations are those of smallest dimension r in a block form.

Matrix representations can have a simple physical interpretation in some cases , such as rotation and reflection matrices . In other cases , such as translation symmetry the symmetry is an invariance on addition of a vector. More general symmetries involve invariance on interchange of various parameters such as electromagnetic fields (e.g. duality).

Groups are also divided into discrete and continuous (or Lie) groups. A lie group has one or more continuous parameters (such as angle, position coordinate ,etc) so that one can think of an infinitessimal change in such parameters as group elements. In the context of the point groups (rotation and reflection), reflection is not a continuous transformation, but proper rotations can be.. So one distinguishes the case of proper rotations which have the determinant of the rotation matrix as +1 by the use of + as a subscript on the group symbol , e.g. O_3^+ .In another common notation this is replaced by the prefix S, e.g. SO_3. Our notation follows the usual physics notation (e.g.[9.8]) with modification to handle our special problems in electromagnetics.

It should be emphasized that our concern here is symmetry , specifically in electromagnetics. Group theory is for our purposes a tool to assist in the exploration for the consequences of various kinds of symmetry that can be exploited for understanding and extending electromagnetic theory and for synthesizing various electromagnetic devices and special field configurations and properties.

III. DUALITY

A fundamental symmetry in the Maxwell equations is that of duality, i.e. the symmetry on interchange of E and H. this is compactly seen by formulating the combined field [2.2,2.5,2.6,2.7,9.1]

$$\mathbf{E}_q(\mathbf{r},t) = \mathbf{E}(\mathbf{r},t) + j\,q\,Z\,\mathbf{H}(\mathbf{r},t) \tag{3.1}$$

$q=\pm 1 \equiv$ separation index

$j = e^{j\pi/2} = \sqrt{-1} \equiv$ unit imaginary

$Z = \sqrt{\frac{\mu}{\varepsilon}}$ $\equiv$ wave impedance of uniform isotropic medium characterized by permeability μ and permittivity ε

Similarly define the combined current density

$$\mathbf{J}_q(\mathbf{r},t) = \mathbf{J}(\mathbf{r},t) + j\frac{q}{Z}\mathbf{J}_m(\mathbf{r},t) \tag{3.2}$$

$\mathbf{J}(\mathbf{r},t) \equiv$ electrical current density

$\mathbf{J}_m(\mathbf{r},t) \equiv$ magnetic current density, in general fictitious but useful as a mathematical artifice as an "equivalent" magnetic current density in various problems.

The various other parameters such as potentials, charge density, Green's functions, etc. can also be defined in combined form [2.8,3.2]. In this combined form the Maxwell equations become

$$\left[\nabla\times - \frac{q}{c}j\frac{\partial}{\partial t}\right]\mathbf{E}_q(\mathbf{r},t) = q\,j\,Z\,\mathbf{J}_q(\mathbf{r},t) \tag{3.3}$$

$$c = \frac{1}{\sqrt{\mu\varepsilon}}$$

In the combined form duality is evident in that interchanging electric and magnetic quantities is equivalent to multiplying the above equations through by $-q\,j$, in which case

$$\mathbf{E}_q^{(d)}(\mathbf{r},t) = -qj\,\mathbf{E}(\mathbf{r},t)$$

$$\mathbf{E}^{(d)}(\mathbf{r},t) = Z\,\mathbf{H}(\mathbf{r},t)\,,\quad \mathbf{H}^{(d)} = -\frac{1}{Z}\mathbf{E}(\mathbf{r},t)$$

$$\mathbf{J}_q^{(d)}(\mathbf{r},t) = -qj\,\frac{1}{Z}\,\mathbf{J}_q(\mathbf{r},t)$$

$$\mathbf{J}^{(d)}(\mathbf{r},t) = \frac{1}{Z}\mathbf{J}_m(\mathbf{r},t)\,,\quad \mathbf{J}_m^{(d)}(\mathbf{r},t) = -Z\mathbf{J}(\mathbf{r},t) \tag{3.4}$$

where the superscript d indicates the dual parameter. Multiplication by qj is a rotation by $q\frac{\pi}{2}$ in the complex plane, this being an element of the C_4 rotation group. Note in addition, however, that one can multiply these equations through by any magnitude 1 complex constant as $e^{j\psi}$ which is equivalent to rotation by ψ in this dual complex plane; taking the real and imaginary parts of the resulting combined complex quantities gives two linear combinations of electric and magnetic quantities which are dual to each other. This kind of rotation is C_∞ or 0_2^+ (a Lie group).

Now there are precisely two ways to form the combined field as given by the choices of q. This can be thought of as two ways of "rotating" E into H. (This rotation aspect will become clearer in the discussion of the Babinet's principle.) The combined field is invariant to transformations in 0_3^+ (rotations only, a Lie group). However, if one includes reflections (giving 0_3, not a lie group this operation of reflection changes q to -q. This is understood in the context of magnetic quantities changing sign under reflection. Later, under the discussion of chirality, the two values of q will be associated with two different propagation constants and polarization rotations.

One application of duality concerns the equivalence principle. As indicated in fig.3.1 the equivalence principle states that for a volume V surrounded by a closed surface S (outward normal $\mathbf{1}_S$) an incident field (no sources in S) can be replaced by surface currents on S (coordinates $\mathbf{r}_S$) giving [2.2]

$$\mathbf{E}_q(\mathbf{r},t) = \begin{cases} \mathbf{E}_q^{(inc)}(\mathbf{r},t) & \text{for } \mathbf{r} \in V \\ 0 & \text{for } \mathbf{r} \notin V \cup S \end{cases} \tag{3.5}$$

$$\mathbf{J}_{S_q}^{(inc)}(\mathbf{r}_s, t) = j\frac{q}{Z}\, \mathbf{1}_S(\mathbf{r}_s) \times \mathbf{E}_q(\mathbf{r}_S,t)$$

$\equiv$ equivalent combined surface current density on S

So given a desired field in V one can construct electric and magnetic surface currents on S to produce this field with no external field. (The roles of inside and outside can also be interchanged). The dual equivalence principle follows with multiplication by qj giving the dual surfaace current densities to produce the dual fields in V. As discussed in [2.3] the electric and magnetic surface current densities can be approximated by distributions of electric and magnetic dipoles on S.

There is also a kind of dual boundary conditions appropriate to quasistatic fields. In this case the fields in V are dual but external fields are allowed and only electric currents are required on the boundary S [2.1,2.2].

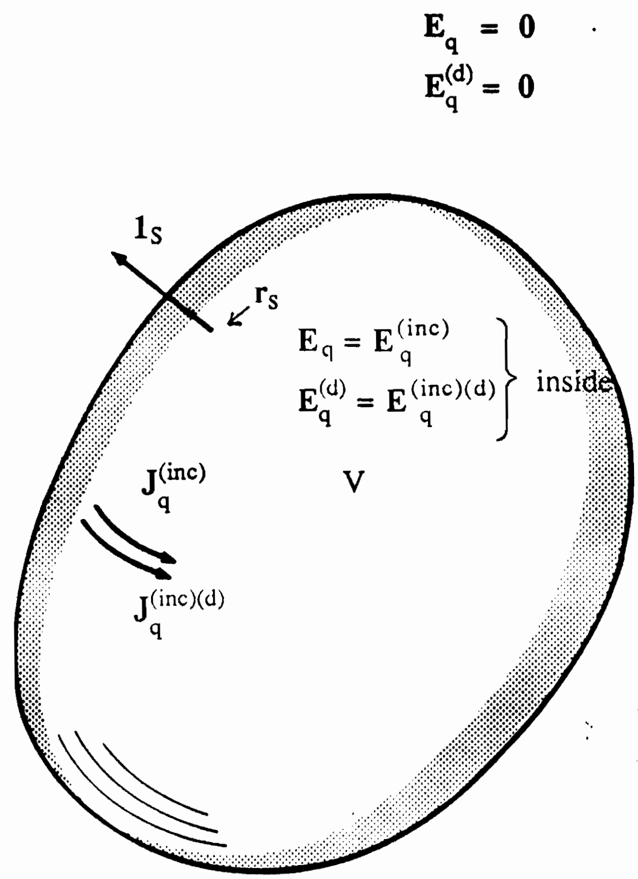


Figure 3.1. Dual Equivalence Principle

IV. RECIPROCITY AND ENERGY THEOREMS

Symmetry is involved in conservation laws, e.g. time reversal symmetry (as in a lossless cavity) implying conservation of electromagnetic energy. There is also a symmetry on interchange (in a general sense) of transmitter and receiver known as reciprocity. These concepts can be combined in a very compact form using the combined field [3.2].

Write the combined Maxwell equation in the complex frequency domain.

$$[\nabla - jq\gamma]\ \tilde{\mathbf{E}}_q(\mathbf{r},s) = j\ q\ Z\ \tilde{\mathbf{J}}_q(\mathbf{r},s) \tag{4.1}$$

$s = \Omega + j\omega \equiv$ Laplace transform variable (complex frequency)

$\sim$ $\equiv$Laplace transform (two sided) , $\gamma = \frac{s}{c} \equiv$ propagation constant

Note that time translation symmetry (a Lie group) is what allows the two-sided Laplace transform to give the above simplification. Using 1 and 2 to denote two different combined fields (with associated combined sources) with in general different values of s and q, let us form (using common relations of vector analysis together with (4.1)) the result

$$\nabla\cdot[\tilde{\mathbf{E}}_{q_1}^{(1)}(\mathbf{r},s_1)\ \times\ \tilde{\mathbf{E}}_{q_2}^{(2)}(\mathbf{r},s_2)\,]$$

$$= [\,\nabla\times\tilde{\mathbf{E}}_{q_1}(\mathbf{r},s_1)]\cdot\ \tilde{\mathbf{E}}_{q_2}(\mathbf{r},s_2)\ -\tilde{\mathbf{E}}_{q_1}(\mathbf{r},s_1)\cdot[\nabla\times\tilde{\mathbf{E}}_{q_2}(\mathbf{r},s_2)]$$

$$= j[q_1\gamma_1 - q_2\,\gamma_2]\,\tilde{\mathbf{E}}_{q_1}(\mathbf{r},s_1)\cdot\tilde{\mathbf{E}}_{q_2}(\mathbf{r},s_2) + j\,q_1\ Z\,\tilde{\mathbf{E}}_{q_2}(\mathbf{r},s_2)\cdot\ \tilde{\mathbf{J}}_{q_1}^{(1)}(\mathbf{r},s_1) - q_2 j\ Z\,\tilde{\mathbf{E}}_{q_1}^{(1)}(\mathbf{r},s_1)\cdot\tilde{\mathbf{J}}_{q_2}^{(2)}(\mathbf{r},s_2) \tag{4.2}$$

Applying the divergence theorem over a volume V with closed boundary surface S gives

$$\int_S [\,\tilde{\mathbf{E}}_{q_1}^{(1)}(\mathbf{r},s_1)\ \times\ \tilde{\mathbf{E}}_{q_2}^{(2)}(\mathbf{r},s_2)\,]\cdot\mathbf{1}_S(\mathbf{r}_s)\ dS$$

$$= \int_V \Big\{ j[q_1\gamma_1 - q_2\gamma_2]\,\tilde{\mathbf{E}}_{q_1}^{(1)}(\mathbf{r},s_1)\cdot\tilde{\mathbf{E}}_{q_2}^{(2)}(\mathbf{r},s_2) + q_1 jZ\tilde{\mathbf{E}}_{q_2}^{(2)}(\mathbf{r},s_2)\cdot\tilde{\mathbf{J}}_{q_1}^{(1)}(\mathbf{r},s_1)$$

$$- q_2 j\,Z\ \tilde{\mathbf{E}}_{q_1}^{(1)}(\mathbf{r},s_1)\cdot\tilde{\mathbf{J}}_{q_2}^{(2)}(\mathbf{r},s_2)]\Big\}\,dV \tag{4.3}$$

This leads to numerous theorems which can be found by appropriate choice of the relation of the "1" and "2" quantities and splitting of the combined quantities. In [3.2] there are numerous results, only some of which are quoted here.

Consider first the set of reciprocity theorems associated with this, defined by

$$q_1 \gamma_1 \equiv q_2 \gamma_2$$

$$\int_S [\, \tilde{\mathbf{E}}^{(1)}_{q_1}(\mathbf{r},s_1) \times \tilde{\mathbf{E}}^{(2)}_{q_2}(\mathbf{r},s_2) \,] \cdot \mathbf{1}_S(\mathbf{r}_s)\, dS$$

$$= j\, Z \int_S [q_1 \tilde{\mathbf{E}}^{(2)}_{q_2}(\mathbf{r},s_2) \cdot \tilde{\mathbf{J}}^{(1)}_{q_1}(\mathbf{r},s_1) - q_2 \tilde{\mathbf{E}}^{(1)}_{q_1}(\mathbf{r},s_1) \cdot \tilde{\mathbf{J}}^{(2)}_{q_2}(\mathbf{r},s_2)] dV \tag{4.4}$$

so that the volume integral contains products of fields and sources (i.e. no fields times fields). This is the same general form as the usual reciprocity theorem [3.17].

The surface integral can be set to zero under certain conditions. Some of these involve cavity problems in which case tangential E is zero on S. here consider the cases that all sources are in V with an outgoing radiation condition. Then let the integral over S be evaluated by that over S_∞ (a surface $\rightarrow \infty$ centered on V), this integral being zero for certain cases. The volume integrals can be conveniently represented in symmetric product form $<\ ,\ >$ implying integration over V. An interesting case has

$$q_1 = q_2 = q \quad , s_1 = s_2 = s$$

$$<\tilde{\mathbf{E}}^{(1)}_q(\mathbf{r},s)\, ;\, \tilde{\mathbf{J}}^{(2)}_q(\mathbf{r},s)> = <\tilde{\mathbf{E}}^{(2)}_q(\mathbf{r},s);\tilde{\mathbf{J}}^{(1)}_q(\mathbf{r},s)> \tag{4.5}$$

$$<\tilde{\mathbf{E}}^{(1)}(\mathbf{r},s);\, \tilde{\mathbf{J}}^{(2)}(\mathbf{r},s)> - <\tilde{\mathbf{H}}^{(1)}(\mathbf{r},s);\tilde{\mathbf{J}}^{(2)}_m(\mathbf{r},s)>$$

$$= <\tilde{\mathbf{E}}^{(2)}(\mathbf{r},s);\tilde{\mathbf{J}}^{(1)}(\mathbf{r},s)> - <\tilde{\mathbf{H}}^{(2)}(\mathbf{r},s);\tilde{\mathbf{J}}^{(1)}_m(\mathbf{r},s)>$$

(electric reciprocity)

$$<\tilde{\mathbf{E}}^{(1)}(\mathbf{r},s);\tilde{\mathbf{J}}^{(2)}_m(\mathbf{r},s)> + Z^2 <\tilde{\mathbf{H}}^{(1)}(\mathbf{r},s);\, \tilde{\mathbf{J}}^{(2)}(\mathbf{r},s)>$$

$$= <\tilde{\mathbf{E}}^{(2)}(\mathbf{r},s);\tilde{\mathbf{J}}^{(1)}_m(\mathbf{r},s)> + Z^2 <\tilde{\mathbf{H}}^{(2)}(\mathbf{r},s);\, \tilde{\mathbf{J}}^{(1)}(\mathbf{r},s)>$$

(magnetic reciprocity)

These symmetric products can be referred to as reaction as introduced by Rumsey [3.17]. Note that what here is referred to as electric reciprocity is the usual form. Setting the magnetic current density to zero simplifies (4.4) and shows the electric reciprocity involves the combination of electric fields and current densities. In addition, however, this type of analysis has shown that there is a magnetic reciprocity involving combinations of magnetic fields and (electric) current densities. Reciprocity is a fundamental symmetry in the Maxwell equations, a symmetry (invariance) on interchange of two solutions.

Second there is the set of energy theorems defined by

$$q_1 \gamma_1 = - q_2 \gamma_2$$

$$\int_S [\tilde{\mathbf{E}}^{(1)}_{q_1}(\mathbf{r},s_1) \times \tilde{\mathbf{E}}^{(2)}_{q_2}(\mathbf{r},s_2)] \cdot \mathbf{1}_S(\mathbf{r}_s)\, dS = j2q_1 \gamma_2 \int_V \tilde{\mathbf{E}}^{(1)}_{q_1}(\mathbf{r},s_1) \cdot \tilde{\mathbf{E}}^{(2)}_{q_2}(\mathbf{r},s_2)\, dV$$

$$+jZ \int_V [q_1\, \tilde{\mathbf{E}}^{(2)}_{q_2}(\mathbf{r},s_2)\cdot \tilde{\mathbf{J}}^{(1)}_{q_1}(\mathbf{r},s_1) - q_2 \tilde{\mathbf{E}}^{(1)}_{q_1}(\mathbf{r},s_1) \cdot \mathbf{J}^{(2)}_{q_2}(\mathbf{r},s_2)]\, dV \tag{4.6}$$

where now we have volume integrals over field products. Like reaction this can be split up into various combinations of fields and currents.

For present purposes let us take the usual case of an outgoing radiation condition. Futhermore, let both solutions be the same in the interesting case

$$q_1 = q_2 = q \ , \ s_2 = -s_2 = s \tag{4.7}$$

$$\int_S [\ \tilde{\mathbf{E}}_q(\mathbf{r},s) \ \times \ \tilde{\mathbf{E}}_q(\mathbf{r},-s)\ ]\, \mathbf{1}_S(\mathbf{r}_s) dS$$

$$= j2q\gamma <\tilde{\mathbf{E}}_q(\mathbf{r},s);\ \tilde{\mathbf{E}}_q(\mathbf{r},-s)> -jqZ[<\tilde{\mathbf{E}}_q(\mathbf{r},s);\tilde{\mathbf{J}}_q(\mathbf{r},-s)> - <\tilde{\mathbf{E}}_q(\mathbf{r},-s);\tilde{\mathbf{J}}_q(\mathbf{r},s)>]$$

This is the usual Poynting vector theorem in combined form; various other forms are derivable from this.

Time-domain forms of these can be found by taking the inverse laplace transform with multiplication going over to convolution. Numerous other forms exist in the literature involving both frequency and time, and other EM quantities such as potentials as well. The present examples exhibit the symmetry between two solutions and the role of duality, this being the purpose here.

V. UNIFORM CHIRAL MEDIA

A chiral object is defined as one which cannot be brought into congruence with its mirror image by any combination of rotation and translation. In the language of the next section (VI) it has no rotation-reflection symmetry S_N. As illustrated in Fig. 5.1 it has a handedness in the usual screw-thread sense. This lack of reflection symmetry can induce some special properties in the Maxwell equations through the constitutive equations as [7.1,7.2,7.3]]

$$\tilde{\mathbf{D}}(\mathbf{r},s)=\tilde{\varepsilon}(s)\tilde{\mathbf{E}}(\mathbf{r},s)+\tilde{\kappa}(s)\tilde{\mathbf{H}}(\mathbf{r},s)$$
$$\tilde{\mathbf{B}}(\mathbf{r},s)=\tilde{\kappa}(s)\tilde{\mathbf{E}}(\mathbf{r},s)+\tilde{\mu}(s)\tilde{\mathbf{H}}(\mathbf{r},s) \qquad (5.1)$$

where $\tilde{\kappa}(s)$ is a measure of the chirality or handedness of the medium. In fig. 5.1 note that a time-varying electric field induces both electric and magnetic moments in the chiral object through the coupling of the straight wire to the loop. A time-varying magnetic field similarly induces both magnetic and electric dipole moments. Assume that there is uniform random distribution of such chiral objects so that we can consider the effect to be represented in an average sense as in (5.1). Note that $\tilde{\varepsilon}(s)$ and $\tilde{\mu}(s)$ are even functions of s if the medium is lossless. Similarly $\tilde{\kappa}(s)$ is an odd function of s (going to zero as $s\rightarrow 0$) as will be seen later.

Combining (5.1) with the Maxwell equations gives

$$\nabla\times\tilde{\mathbf{E}}(\mathbf{r},s) =-s\tilde{\mathbf{B}}(\mathbf{r},s) = s[\tilde{\kappa}(s)\tilde{\mathbf{E}}(\mathbf{r},s)-\tilde{\mu}(s)\tilde{\mathbf{H}}(\mathbf{r},s)]$$
$$\nabla\times\tilde{\mathbf{H}}(\mathbf{r},s) = s\tilde{\mathbf{D}}(\mathbf{r},s) = s[\tilde{\varepsilon}(s)\tilde{\mathbf{E}}(\mathbf{r},s)+\tilde{\kappa}(s)\tilde{\mathbf{H}}(\mathbf{r},s)] \qquad (5.2)$$

Now form a combined field analogous to (3.1) giving (with no sources in the region of concern)

$$\tilde{\mathbf{E}}_q(\mathbf{r},s) \equiv \tilde{\mathbf{E}}(\mathbf{r},s) +\tilde{\zeta}_q(s)\tilde{\mathbf{H}}(\mathbf{r},s)$$
$$\nabla\times\tilde{\mathbf{E}}_q(\mathbf{r},s) = s\left\{ [\tilde{\kappa}(s)+\tilde{\zeta}_q(s)\tilde{\varepsilon}(s)]\, \tilde{\mathbf{E}}(\mathbf{r},s)+[\tilde{\kappa}(s)\tilde{\zeta}_q(s)-\tilde{\mu}(s)]\, \tilde{\mathbf{H}}(\mathbf{r},s) \right\}$$
$$\equiv s[\tilde{\kappa}(s)+\tilde{\zeta}_q(s)\tilde{\varepsilon}(s)]\tilde{\mathbf{E}}_q(\mathbf{r},s) \qquad (5.3)$$
$$\tilde{\zeta}_q(s) = \frac{\tilde{\kappa}(s)\tilde{\zeta}_q(s)-\tilde{\mu}(s)}{\tilde{\kappa}(s)+\tilde{\zeta}_q(s)\tilde{\varepsilon}(s)}$$

This is solved as

$$\tilde{\zeta}_q(s) = qj\sqrt{\frac{\tilde{\mu}(s)}{\tilde{\varepsilon}(s)}} \equiv qj\, \tilde{Z}(s)\ ,\ q=\pm 1 \qquad (5.4)$$

so that the same simple form as in (3.1) is possible . In complex frequency domain the combined Maxwell equation with no sources takes the form

$$[\nabla\times -qj\, \tilde{\gamma}_q(s)]\tilde{\mathbf{E}}_q(\mathbf{r},s) = \mathbf{0}$$
$$\tilde{\gamma}_q(s)\equiv s\, [\sqrt{\tilde{\mu}(s)\tilde{\varepsilon}(s)}\ - qj\tilde{\kappa}(s)]^{-1}$$
$$\equiv \frac{s}{\tilde{v}_q(s)} \qquad (5.5)$$
$$\tilde{v}_q(s) = [\sqrt{\tilde{\mu}(s)\tilde{\varepsilon}(s)}\ - qj\tilde{\kappa}]^{-1}\equiv \text{phase speed}$$

For a right-handed medium (fig 5.1) $\tilde{\kappa}(j\omega)$ is negative imaginary, and conversely for a left-handed medium. Then for a right-handed medium the chirality increases $\tilde{v}_+$ and decreases $\tilde{v}_-$ (and conversely for a left-handed medium), thereby giving two different wave speeds in

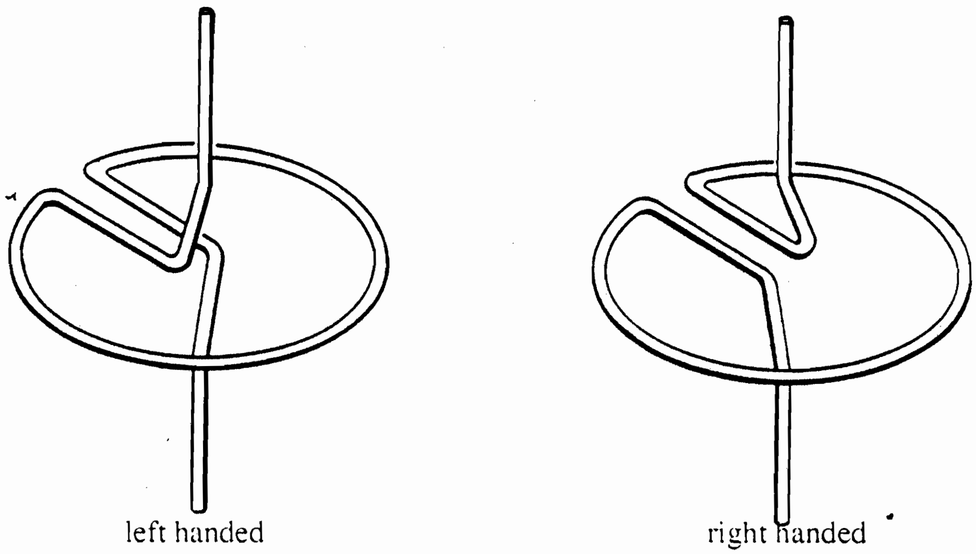


Figure 5.1 Chiral Object Linking Electric and Magnetic Terms of Average Constitutive Relations.

the medium. Note that for $s=j\omega$ real $\tilde{\mu}(s)$ and $\tilde{\varepsilon}(s)$ require imaginary $\tilde{\kappa}$ for the speed to be real and give non-attenuating (lossless)wave propagation.

Consider an arbitrary plane wave which ,since the medium is isotropic ,we can take as propagating in the z direction as

$$\tilde{\mathbf{E}}_q(\mathbf{r},s) = E_0\sqrt{2}\,\mathbf{1}_q e^{-\tilde{\gamma}_q(s)z}$$

$$\mathbf{1}_q = \frac{1}{\sqrt{2}}[\mathbf{1}_x + qj\,\mathbf{1}_y] \tag{5.6}$$

which is consistent with (5.5). Using standard conventions [4.4,4.5] then + gives left-handed polarization (LHP) and - gives RHP, both circular of course.

This business of chirality can then be looked at as one of breaking symmetry . A non-chiral isotropic medium has identical properties for both RHP and LHP waves. A chiral medium gives these different speeds (or propagation constants). Viewed another way reversing a space coordinate or time (a reflection) in a lossless medium is equivalent to reversing the magnetic field and hence the sign of q. The handedness of the medium destroys this mirror symmetry. In terms of three space coordinates then a chiral medium has O_3^+ symmetry (proper rotations , a Lie group) but not O_3 symmetry (which includes spatial reflections).

VI. POINT SYMMETRY GROUPS : ROTATIONS AND REFLECTIONS

The point symmetry groups consist of rotations, reflections, and combinations thereof. Under these transformations at least one point (the center of symmetry) remains fixed. These groups are very important for symmetry in scattering and the design of various electromagnetic devices. See [9.8] for a thorough discussion of these groups. Here we focus on their significance in electromagnetics.

Begin with C_N, an N-fold rotation axis $\mathbf{1}_o$. Considering as a rotation angle (right handed) about this axis we have the commutative group (order N)

$$C_N = \{(C_N)_\ell \mid \ell=1,2, \ldots,N\}$$

$$(C_N)_\ell \equiv \text{rotation by } \frac{2\pi\ell}{N} = (C_N)_1^\ell$$

$$(C_N)_1^N = (C_N)_N = (1) \qquad \text{(closure)} \tag{6.1}$$

This has a matrix representation

$$(C_{n,m}(\phi_\ell))=\begin{pmatrix}\cos(\phi_\ell) & -\sin(\phi_\ell)\\ \sin(\phi_\ell) & \cos(\phi_\ell)\end{pmatrix} = \exp\left(\phi_\ell\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right)$$

$$\phi_\ell = \frac{2\pi\ell}{N} \quad , \ell = 1,2,\ldots, N \tag{6.2}$$

$$(C_{n,m}(0)) = (C_{n,m}(2\pi)) = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

One can also represent this group in scalar form by a complex rotation $e^{j\phi_\ell}$ but the above form is better for what comes later.

C_N symmetry is related also to problems involving circulant matrices[4.15]. Considering some appropriate matrix such as an impedance matrix $(\tilde{Z}_{n,m}(s))$ (or an admittance matrix),then if this corresponds to an electromagnetic structure such as a circular antenna array in fig.6.1 with C_N symmetry [4.10,4.11], the matrix elements have the property

$$\tilde{Z}_{n,m}(s) = \text{function of } m-n+MN \tag{6.3}$$
$$M = \text{any integer}$$

i.e. these are only functions of the differences in the angular position of the elements , ϕ_m-ϕ_n around the array. Such matrices have eigenvectors $\frac{1}{\sqrt{N}}(e^{j2\pi\beta n/N})$ for β=1,2,..N.

However we usually have reciprocity in our electromagnetic problems (section IV), i.e.

$$(\tilde{Z}_{n,m}(s)) =(\tilde{Z}_{n,m}(s))^T$$
$$\tilde{Z}_{n,m}(s) = \tilde{Z}_{m,n}(s) = \text{function of } |m-n| \text{ or } |\phi_m-\phi_n| \tag{6.4}$$

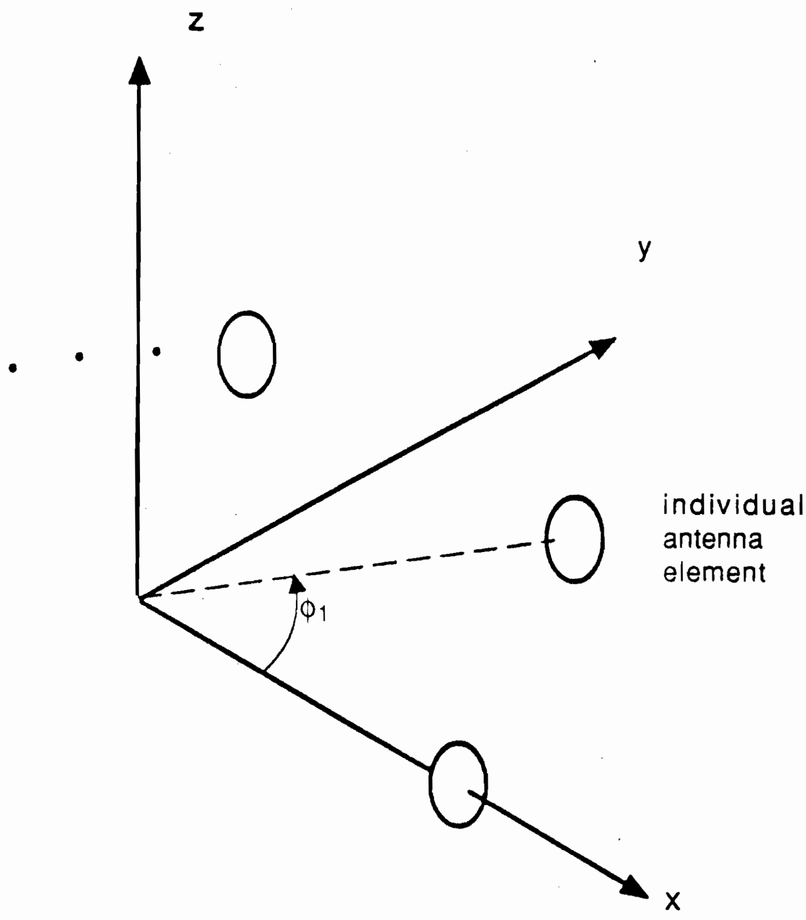


Figure 6.1 Circular Antenna Array.

Such symmetric circular matrices are called bicirculant. The extra condition increases the symmetry [4.15] allowing the left and right eigenvectors to be the same and the eigenvectors to be expressible as real vectors (doubly degenerate) by taking real and imaginary parts (cos and sin). Furtheremore the number of eigenvalues is reduced as

$$\text{number of eigenvalues}=\begin{cases}\frac{N}{2} \text{ for N even}\\ \frac{N+1}{2} \text{ for N odd}\end{cases} \tag{6.5}$$

Due to these additional properties beyond purely C_N symmetry one can define reciprocal rotation symmetry (uniaxial) as C_{N_r}. Note that in some cases the bicirculant matrices are real (as in a multiconductor transmission line with wires uniformly spaced around a circle) in which case the associated eigenvalues are all real (as well as positive).

Such uniaxial symmetry can appear in many objects. For present illustration consider on-axis backscattering as in figure 6.2 with radar polarizations $\mathbf{1}_h$ (horizontal) and $\mathbf{1}_v$ ("vertical" actually in a vertical plane).As shown in [4.4,4.5] the backscattering matrix takes the form

$$\tilde{\mathbb{c}}_b(s) =\tilde{c}_b(s) \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \text{ for } N\geq 3 \tag{6.6}$$

i.e. a scalar function of frequency times the identity. Said another way for $N\geq3$ there is no depolization of the on-axis backscattered signal. Note that no symmetry planes have been assumed for this purpose. However, reciprocity is important so as to constrain the matrix (backscattering) to be symmetric. A two-fold axis (fig. 6.2C) is not sufficient as can be seen by alligning the incident polarization along the assumed two symmetry planes defined by ψ_0 and $\psi_0 + \frac{\pi}{2}$. Going to $N\geq3$ successive rotation by $\frac{2\pi}{N}$ gives two linearly independent vectors to span the two-dimensional space, these two orientations as polarizations giving identical backscattering characteristics by the symmetry. One can extend C_N as $N\rightarrow\infty$ so that a body of revolution (fig.6.2A) is obtained. A C_∞ body of revolution need not have a symmetry plane; it might consist, for example, of anisotropic materials such as a large number (as $N\rightarrow\infty$) of conducting spiral (non-radial) spokes; let N=3 in fig. 6.2 B be altered by increasing N arbitrarily. Note C_∞ is a Lie group also known as O_2^+ (orthogonal group of proper rotations in two dimensions (no reflections)).

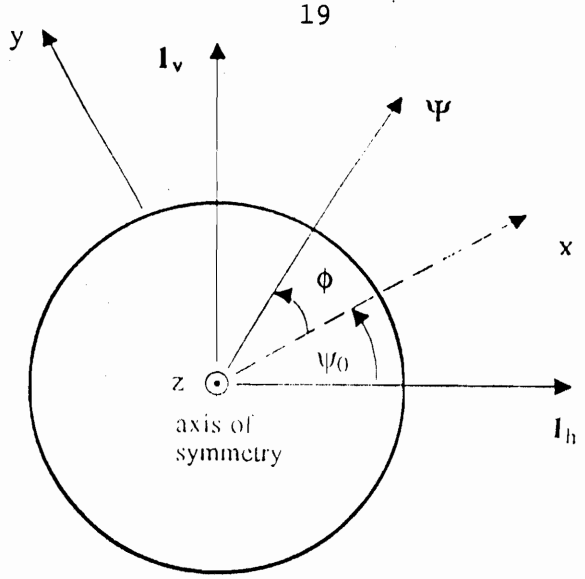
Considering reflections we have the group (order 2)

$$R= \{ (1),(R) \},\ (R)^2 =(1) \tag{6.7}$$

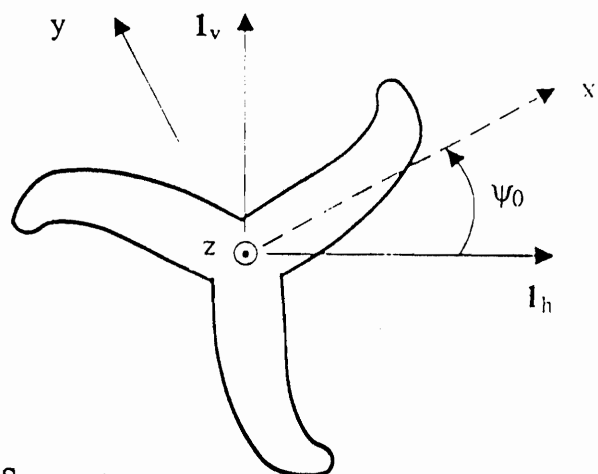
with matrix representation

$$\mathbb{R} = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix} \tag{6.8}$$

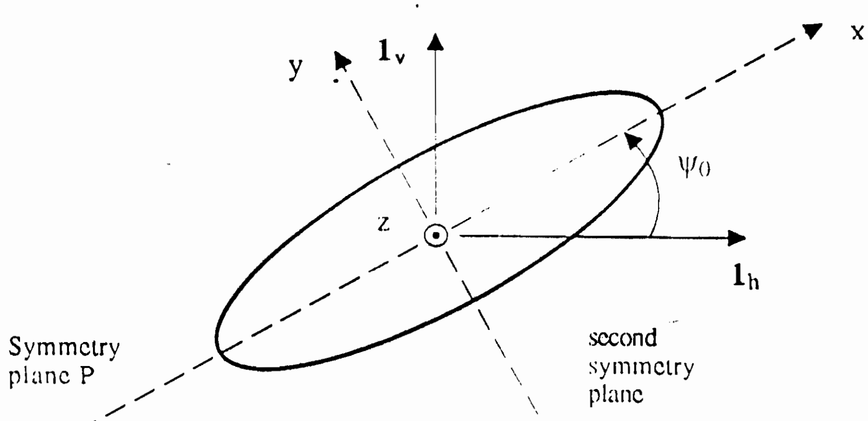
where in a three-dimensional space this is equivalent to reversing one axis, typically taken as the z axis when not combined with other symmetries.



A. Body of Revolution



B. C_3 Symmetry



C. C_{2a} Symmetry

Figure 6.2 On-Axis Backscattering

Such reflection symmetry is very common. To illustrate this consider a typical aircraft with a vertical symmetry plane, as in figure 6.3 As discusssed in [4.1,4.2,4.7] reflection symmetry decomposes the response into two parts denoted symmetric and antisymmetric, which (for the surface current density vector) reflect through the symmetry plane (multiplication by $\mathbb{R}$) with + and - signs respectively. By positioning a sensor on the symmetry plane P one can orient it to be sensitive to only one of these two parts of the field, thereby eliminating one entire set of natural modes in the scattered field received by the sensor. The interesting case has a magnetic sensor on P oriented parellel to P. This eliminates the symmetric part of the aircraft scattering (including the major fuselage modes) from the sensor response. The position and orientation of the magnetic sensor on P are then chosen to minimize sensitivity to the antisymmetric wing and horizontal-stabilizer resonances as well as low-frequency distortion near the fuselage. As indicated in fig. 6.3 this leads to two optimal choices designed by positions $\mathbf{r}_1$ and $\mathbf{r}_2$ on P, on a line through the projections of the wing and horizontal stabilizer on P, and with orientation (sensor axis of sensitivity) parallel to this line.

Adjoining reflections with rotations, first let (R_a) denote reflection through a symmetry plane containing an N-fold rotation axis giving the group (order 2N)

$$C_{Na} = \{(C_N)_\ell \,, (R_a)(C_N)_\ell \mid \ell = 1,2,\dots,N\} \tag{6.9}$$

There are N axial symmetry planes in this kind of symmetry. This turns out to be an important kind used for antenna arrays, such as for direction finders. In such a case one might have N elementary antennas (say finite-length wires) evenly spaced around a circle on a plane perpendicular to $\mathbf{1}_0$. with the currents parellel to $\mathbf{1}_0$. In this case by adjusting the currents as $\cos(p\phi_\ell+\psi_0)$ one can excite specific antenna patterns associated with each relevant integer p. One can steer the beam and have narrow, well-defined nulls (useful for locating a scatterer (or radiator) with respect to azimuth ϕ).Appendix C considers such an example in detail .

One can also form C_{Nt} by adjoining R_t meaning reflection through a plane transverse to the N-fold axis. Rotation-reflection symmetry S_N is defined by

$$S_N = \{(S_N)_\ell \mid \ell = 1,2,\dots,N\}$$

$$(S_N)_1 = (C_N)_1 (R_t) = (R_t)(C_N)_1 \tag{6.10}$$

$$(S_N)_\ell = (S_N)_2^\ell = (C_N)_1^\ell (R_t)^\ell = (C_N)_\ell (R_t)^\ell$$

which applies particularly for N even, since for odd N this becomes equivalent to C_{Nt}[9.8].

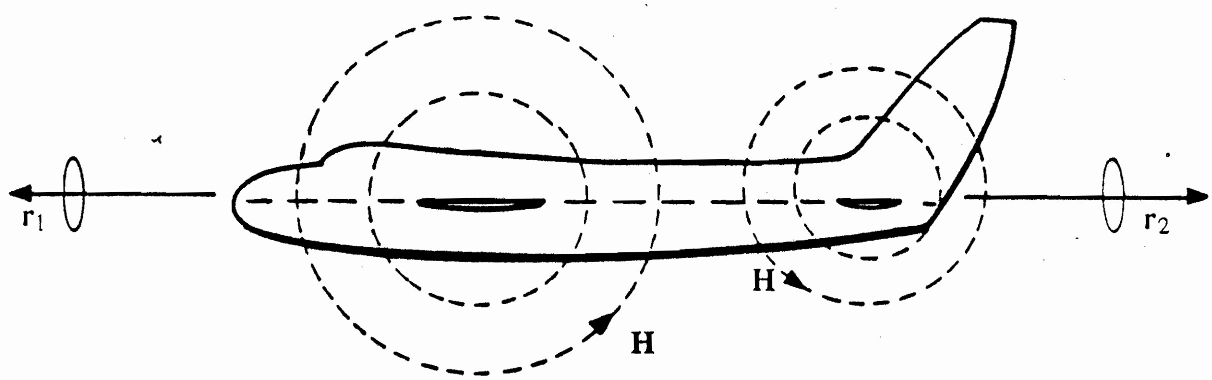


Figure 6.3 A special location and orientation for the magnetic sensor to minimize scattering from both wing and horizontal stabilizer.

Higher-order rotation symmetry is first dihedral symmetry defined by adjunction of a 2-fold axis to a principal N-fold axis given symbolically by (2N elements)

$$D_N = \{ (C_N)_\ell \, , \, (C_N)_\ell \, (C_2^{(1)}) \mid \ell=1,2,\ldots,N \} \tag{6.11}$$

where $C_2^{(1)}$ is rotation by about an axis $\mathbf{1}_1$ which is perpendicular to the N-fold axis $\mathbf{1}_o$. This generates a set of N 2-fold axes which can be designated $\mathbf{1}_u$ for u=1,2,...,N. Note that this symmetry involves only proper rotations $(C_N)_\ell$ and $(C_2^{(u)})$, i.e. no reflections.

An illustration of dihedral symmetry is given in fig. 6.4 for a special kind of high-frequency capacitor [4.8]. Here the principal axis $\mathbf{1}_o$ is taken as 2-fold and there are two other C_2 axes at right angles to it. This particular example has an equal number of foils in each of two sets (A and B) in such a way that the tabs and foils are the same piece of metal (rectangular). The separating dielectric sheets would in general be square. In this example there are two more symmetry planes P_1 and P_2 spaced between the secondary C_2 axes. These are special axial (a) planes referred to as diagonal (d) planes, giving D_{2d} symmetry (order 8). Note that the tabs are bent "up" (A) and"down" (B) to connect to the next capacitors in a column and that the column itself retains dihedral symmetry. One can go to even higher N for such capacitors by using, say circular foils with N tabs with positions alternating for A and B.

Considering more than one N-fold axis with N>2 we have the groups associated with the regular polyhedra : T (tetrahedral);O (octahedral, including the cube), and Y (icosahedral, including the dodecahedron). To these one can adjoin symmetry planes with "d" or "t" reflection as in the case of D_N. An interesting application of this kind of symmetry would be for an antenna array which would steer a beam with two angles to specify a direction, i. e.,θ and ϕ in spherical coordinates (useful for locating a scatterer (or radiator) with respect to both angles.

Finally, we have the Lie group O_3^+ consisting of continuous rotations with respect to two angles. This is visualized by considering some point on the unit sphere. Choose some direction at this point (an angle) and advance some angle around the sphere. This can be done from every point on the unit sphere. Adjoining reflections via planes through the center of symmetry (all such planes) gives O_3 (not a Lie group) which keeps $x^2+y^2+z^2$ invariant under the symmetry operations.

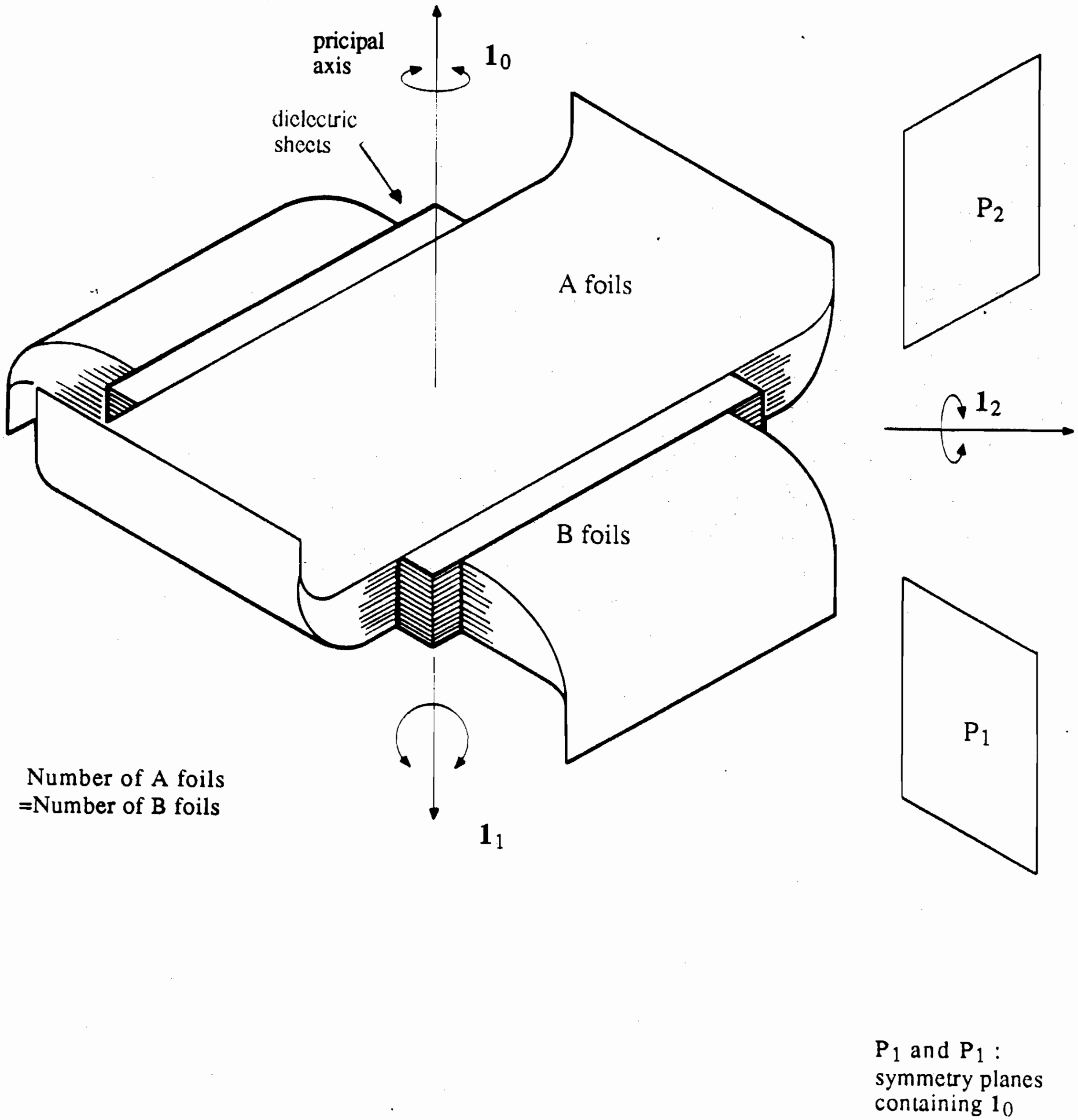


Figure 6.4 Dihedral D_{2d} Capacitor

VII. GENERALIZED BABINET PRINCIPLE (C_{Nc} Symmetry)

Now combine the duality symmetry in Maxwell equations with a geometrical symmetry , specifically C_N symmetry in a planar structure . First consider ,however, what is called a complementary structure (planar) in formulating the Babinet's principle [6.3].

Multiplying the combined field in (3.1) by -qj to give the dual combined field we form the dual fields as in (3.4). Consider the z=0 plane , designated S, as a plane on which electric surface current density and various sheet impedances (including perfect sheet conductors and sheet insulators) are assumed to exist. Then assume that the complementary fields are the dual fields for z>0 but the negative of the dual fields for z<0. Here we assume that sources (for simplicity) are on S and waves are outgoing [6.2]. This allows us to take the original and complementary fields as symmetric with respect to S as discussed in section VI [4.1,4.2,4.7].

Letting the tangential fields on S be designated by a subscript s and letting them be considered as two-dimensional quantities with cylindrical (Ψ,ϕ) coordinates we have using the fields at $z=0^+$,

$$\begin{aligned}\mathbf{E}_s^{(d)}(\Psi,\phi;t) &= Z\,\mathbf{H}_s(\Psi,\phi;t)\\ \mathbf{H}_s^{(d)}(\Psi,\phi;t) &= -\frac{1}{Z}\,\mathbf{E}_s(\Psi,\phi;t)\end{aligned} \tag{7.1}$$

Applying boundary conditions on S for the magnetic field gives

$$\mathbf{J}_s(\Psi,\phi;t) = 2\,\mathcal{T}_d\cdot\mathbf{H}_s(\Psi,\phi;t) \qquad \text{(original surface current density)}$$

$$\mathbf{J}_s^{(c)}(\Psi,\phi;t) = 2\,\mathcal{T}_d\cdot\mathbf{H}_s^{(c)}(\Psi,\phi;t) \qquad \text{(complementary surface current density)}$$

$$\mathcal{T}_d = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \frac{\pi}{2} \text{ rotation} \tag{7.2}$$

Note the complementary surface current density (not dual as in (3.4)), so that we deal with only electric surface current densities (both original and complementary). Let regions of S as appropriate be characterized by a sheet admittance $\tilde{\mathbb{Y}}_s$ (a dyadic or matrix , 2X2) for electric boundary conditions (away from sources) as

$$\begin{aligned}\tilde{\mathbf{J}}_s(\Psi,\phi;s) &= \tilde{\mathbb{Y}}_s(\Psi,\phi;s)\cdot\tilde{\mathbf{E}}_s(\Psi,\phi;s)\\ \tilde{\mathbf{J}}_s^{(c)}(\Psi,\phi;s) &= \tilde{\mathbb{Y}}_s^{(s)}(\Psi,\phi;s)\cdot\tilde{\mathbf{E}}_s^{(d)}(\Psi,\phi;s)\end{aligned} \tag{7.3}$$

this also defining the complementary sheet admittance . One includes incident fields (not symmetric with respect to S in general) by adding to (7.3) (on the right side) sources of the form

$$\tilde{\mathbf{J}}_s^{(s)}(\Psi,\phi;s) = \tilde{\mathbb{Y}}_s(\Psi,\phi;s)\cdot\tilde{\mathbf{E}}^{(inc)}(\Psi,\phi,z;s)\Big|_{z=0}$$

$$\tilde{\mathbf{J}}_s^{(c,s)}(\Psi,\phi;s) = \tilde{\mathbb{Y}}_s^{(c)}(\Psi,\phi;s)\cdot\tilde{\mathbf{E}}^{(d,inc)}(\Psi,\phi,z;s)\Big|_{z=0} \tag{7.4}$$

In this case $\tilde{\mathbf{E}}_s(\Psi,\phi;s)$ and $\tilde{\mathbf{E}}_s^{(d)}(\Psi,\phi;s)$ in (7.3) are associated with the scattered field (symmetric) , which when added to the incident fields as in (7.4) give the total fields. Alternatively ,one can use the magnetic field (and its dual) to define these sources via (7.2) for source regions on S , such as the "gap" region for an antenna problem.

Combining (7.1) through (7.3) we have

$$\tilde{\mathbf{J}}_s^{(s)}(\Psi,\phi;s) = \tilde{\mathbb{Y}}_s^{(c)}(\Psi,\phi;s)\cdot\tilde{\mathbf{E}}_s^{(d)}(\Psi,\phi;s) = 2\,\mathcal{T}_d\cdot\tilde{\mathbf{H}}_s^{(d)}(\Psi,\phi;s)$$
$$= -\frac{2}{Z}\,\mathcal{T}_d\cdot\tilde{\mathbf{E}}_s(\Psi,\phi;s) = -\frac{2}{Z}\,\mathcal{T}_d\cdot\tilde{\mathbb{Y}}_s^{-1}(\Psi,\phi;s)\cdot\tilde{\mathbf{J}}_s(\Psi,\phi;s)$$
$$= -\frac{4}{Z}\,\mathcal{T}_d\cdot\tilde{\mathbb{Y}}_s^{-1}\cdot\mathcal{T}_d\cdot\tilde{\mathbf{H}}_s(\Psi,\phi;s) = -\frac{4}{Z^2}\,\mathcal{T}_d\cdot\tilde{\mathbb{Y}}_s^{-1}\cdot\mathcal{T}_d\cdot\tilde{\mathbf{E}}_s^{(d)}(\Psi,\phi;s) \tag{7.5}$$

with inverse here taken in the two dimensional sense. Noting that

$$\mathcal{T}_d^{T} = \mathcal{T}_d^{-1} = -\,\mathcal{T}_d = \begin{pmatrix} 0 & 1 \\ -1 & -0 \end{pmatrix} = -\frac{\pi}{2}\ \text{rotation} \tag{7.6}$$

and letting $\tilde{\mathbf{E}}_s^{(d)}(\psi,\phi;s)$ be arbitrary we have

$$\tilde{\mathbb{Y}}_s^{(c)}(\Psi,\phi;s) = -\frac{4}{Z^2}\,\mathcal{T}_d\cdot\tilde{\mathbb{Y}}_s^{-1}\cdot\mathcal{T}_d^{T} \tag{7.7}$$

as an explicit formula for the complementary sheet admittance [6.1,6.7] . In normalized form define

$$\tilde{\mathbf{y}}_s(\Psi,\phi;s) \equiv -\frac{Z}{2}\,\tilde{\mathbb{Y}}_s(\Psi,\phi;s)\ ,\ \ \tilde{\mathbf{y}}_s^{(c)}(\Psi,\phi;s) \equiv -\frac{Z}{2}\,\tilde{\mathbb{Y}}_s^{(c)}(\Psi,\phi;s) \tag{7.8}$$

giving

$$\tilde{\mathbf{y}}_s^{(c)}(\Psi,\phi;s) = \mathcal{T}_d\cdot\tilde{\mathbb{Y}}_s^{-1}\cdot\mathcal{T}_d^{T} \tag{7.9}$$

as a convenient form of the result . So on S the complementary admittance is a rotated inverse. This is consistent with the usual Babinets principle as can be seen by considering a scalar admittance sheet as

$$\tilde{\mathbb{Y}}_s(\Psi,\phi;s) = \tilde{Y}_s(\Psi,\phi;s)\,\mathbb{1}_t\ ,\ \tilde{\mathbb{Y}}_s^{(c)}(\Psi,\phi;s) = \tilde{Y}_s^{(c)}(\Psi,\phi;s)\,\mathbb{1}_t$$
$$\tilde{y}_s(\Psi,\phi;s) = \frac{Z}{2}\,\tilde{Y}_s(\Psi,\phi;s)\ ,\ \ \tilde{y}_s^{(c)}(\Psi,\phi;s) = \frac{Z}{2}\,\tilde{Y}_s^{(c)}(\Psi,\phi;s) \tag{7.10}$$

$$\mathbb{1}_t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{1}_x\mathbf{1}_x + \mathbf{1}_y\mathbf{1}_y \equiv \text{two dimensional (transverse) identity}$$
$$= -\,\mathcal{T}_d^{2} = \mathcal{T}_d^{4}$$

in which case (7.9) becomes

$$\tilde{y}_s^{(c)}(\Psi,\phi;s) = \tilde{y}_s^{-1}(\Psi,\phi;s) \tag{7.11}$$

A perfectly conducting sheet has $\tilde{Y}_S = \infty$ so that the compliment has $\tilde{Y}_S^{(c)} = 0$,i.e. an open aperture (and conversely). The form in (7.9) is ,however, considerably more general. In effect this defines the complementary "geometry".

Having found the complementary problem on S let us go on to the self-complementary problem . This is usually defined by requiring that the complementary problem be the same as the original problem after a rotation in S about some point which we take as the coordinate origin Ψ=0. Letting the rotating angle be ϕ_c we have

$$\tilde{y}_s(\Psi,\phi+\phi_c\,;s) = \mathcal{T}_c\cdot\tilde{y}_s^{(c)}(\Psi,\phi+\,\phi_c\,;s)\cdot\mathcal{T}_c^T$$

$$\mathcal{T}_c = \begin{pmatrix} \cos(\phi_c) & -\sin(\phi_c) \\ \sin(\phi_c) & \cos(\phi_c) \end{pmatrix} \equiv \text{rotation by } \phi_c \tag{7.12}$$

$$\mathcal{T}_c^{-1} = \mathcal{T}_c^T \equiv \text{rotation by} - \phi_c$$

Substituting from (7.9) gives the basic self-complementary relationship as

$$\tilde{y}_s(\Psi,\phi+\phi_c\,;s) = \mathcal{T}_c\,.\,\mathcal{T}_d.\,\tilde{y}_s^{-1}(\Psi,\phi;s)\;\mathcal{T}_d^T.\mathcal{T}_c^T \tag{7.13}$$

Applying this twice (an additional rotation by ϕ_c gives

$$\tilde{y}_s(\Psi,\phi+2\phi_c\,;s) = \mathcal{T}_c\cdot\mathcal{T}_d\cdot\,\tilde{y}_s^{-1}(\Psi,\phi+\phi_c;s)\;\mathcal{T}_d^T\cdot\mathcal{T}_c^T$$

$$= \mathcal{T}_c^2\cdot\,\tilde{y}_s(\Psi,\phi;s)\,\cdot\,\mathcal{T}_c^{-2} \tag{7.14}$$

using the commutativity of the rotation matrices. This shows a periodicity of $2\phi_c$ Requiring periodicity of 2π for the sheet admittance requires

$$2\phi_c N = 2\pi\ ,\ N=1,2,... \tag{7.15}$$

which defines C_N symmetry ,i.e. invariance on rotation by $\frac{2\pi}{N}$. Noting that S itself is a transverse symmetry plane , one might consider this a case of higher symmetry ,except that it is degenerate since such a reflection is only an identity for all points on S.

Now we are in a position to define the self-complementary rotation group (order 2N) by

$$C_{Nc} = \left\{ (C_N)_\ell , (C_N)_\ell (C_{Nc}) \mid \ell = 1,2...,N \right\}$$

$$= \left\{ (C_{Nc})^\ell \mid \ell = 1,2...,N \right\} \tag{7.16}$$

$C_{Nc} \equiv$rotation by ϕ_c $(=\frac{\pi}{N})$ and taking of complement as in (7.9)

$(C_{Nc})^2 = (C_N)_1$

This is a commutative group. The basic group element has an operator representation

$$(C_{Nc}) \rightarrow \mathcal{T}_c\cdot\mathcal{T}_d\cdot(\quad)^{-1}\cdot\,\mathcal{T}_d^T\cdot\,\mathcal{T}_c^T$$

$$(C_{Nc}) \rightarrow \mathcal{T}_c^2\cdot\;(\quad)\cdot\;\mathcal{T}_c^{-2} \tag{7.17}$$

Note the inclusion of matrix inversion (not just multiplication) in the representation . One can also impose other symmetries such as R_a (axial symmetry planes placed ϕ_c apart) which is equivalent to 2-fold rotation axes in S giving dihedral symmetry D_N which is combined with the self-complementary property to give D_{Nc} symmetry (or C_{Nca} symmetry) of order 4N.

For scalar sheet admittances the self-complementary relationship (7.13) becomes

$$\tilde{y}_s(\Psi,\phi+\phi_c\,;s) = \tilde{y}_s^{-1}(\Psi,\phi+\phi_c\,;s) \tag{7.18}$$

As one varies ϕ around the circle the normalized admittance is inverted after each change by ϕ_c. For typical self-complementary antennas this is an alternative between 0 (nothing) and ∞ (perfect conductor) [8.1]. For dyadic sheet admittances assume the realizable diagonal form

$$\tilde{\mathbf{y}}_s(\Psi,\phi\,;s) = \tilde{y}_{s_a}(\Psi,\phi\,;s)\mathbf{1}_a(\phi)\mathbf{1}_a(\phi) + \tilde{y}_{s_b}(\Psi,\phi\,;s)\mathbf{1}_b(\phi)\mathbf{1}_b(\phi)$$

$$\mathbf{1}_b(\phi) = \mathcal{T}_d\cdot\mathbf{1}_a(\phi) \quad , \mathbf{1}_a\ ,\mathbf{1}_b \text{ real} \tag{7.19}$$

$$\tilde{\mathbf{y}}_s^{-1}(\Psi,\phi\,;s) = \tilde{y}_{s_a}^{-1}(\Psi,\phi\,;s)\mathbf{1}_a(\phi)\mathbf{1}_a(\phi) + \tilde{y}_{s_b}^{-1}(\Psi,\phi\,;s)\mathbf{1}_b(\phi)\mathbf{1}_b(\phi)$$

Now the self-complementary relationship (7.13) gives

$$\tilde{\mathbf{y}}_s(\Psi,\phi+\phi_c;s) = \tilde{y}_{s_a}^{-1}(\Psi,\phi\,;s)\mathbf{1}_b(\phi+\phi_c)\mathbf{1}_b(\phi+\phi_c) + \tilde{y}_{s_b}^{-1}(\Psi,\phi;s)\mathbf{1}_a(\phi+\phi_c)\mathbf{1}_a(\phi+\phi_c) \tag{7.20}$$

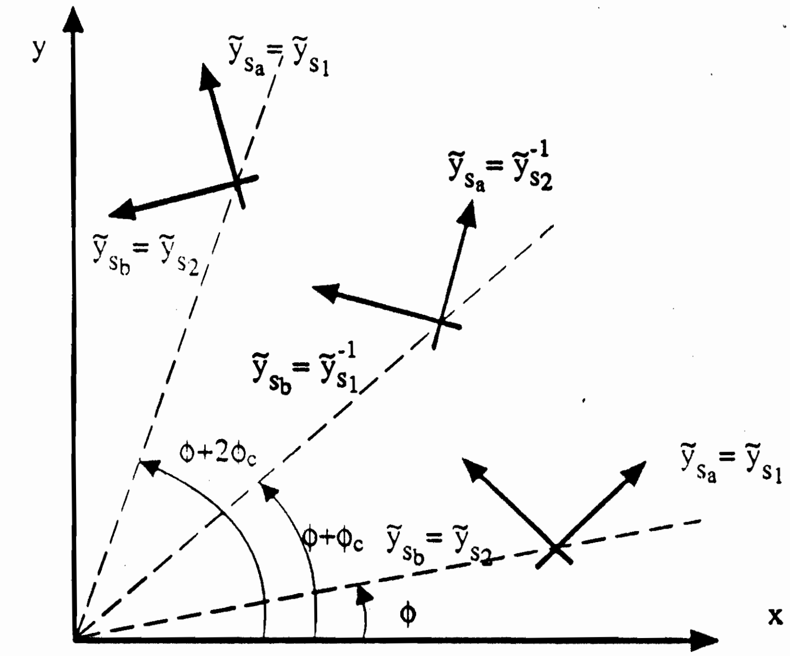
Referring to fig.7.1A what is happening is that the unit vectors $\mathbf{1}_a$ and $\mathbf{1}_b$ are rotated by ϕ_c for each advance in ϕ by ϕ_c .However, the corresponding eigenvalues (eigenadmittances) are not only inverted , but interchanged in advance by ϕ_c . Labelling $\tilde{y}_{s_a}$ and $\tilde{y}_{s_b}$ by $\tilde{y}_{s_1}$ and $\tilde{y}_{s_2}$ at some starting ϕ , they take on values $\tilde{y}_{s_2}^{-1}$ and $\tilde{y}_{s_1}^{-1}$, respectively , on advance by ϕ_c , returning to $\tilde{y}_{s_1}$ and $\tilde{y}_{s_2}$, on a second advance by ϕ_c

A special case of this concerns an added symmetry which occurs in the scalar case (7.18) if the normalized admittance is 1. Referring to this situation as self inverse note that there is no change on advancement by ϕ_c. If this condition is maintained throughout S, this is an impedance sheet of value $\frac{Z}{2}$, a case of C_∞ or D_∞ symmetry. For the dyadic case as illustrated in Fig. 7.1B one has what can be termed a self rotated inverse , that is on advancement by ϕ_c , $\tilde{\mathbf{y}}_s$ is merely rotated by ϕ_c with the constraint

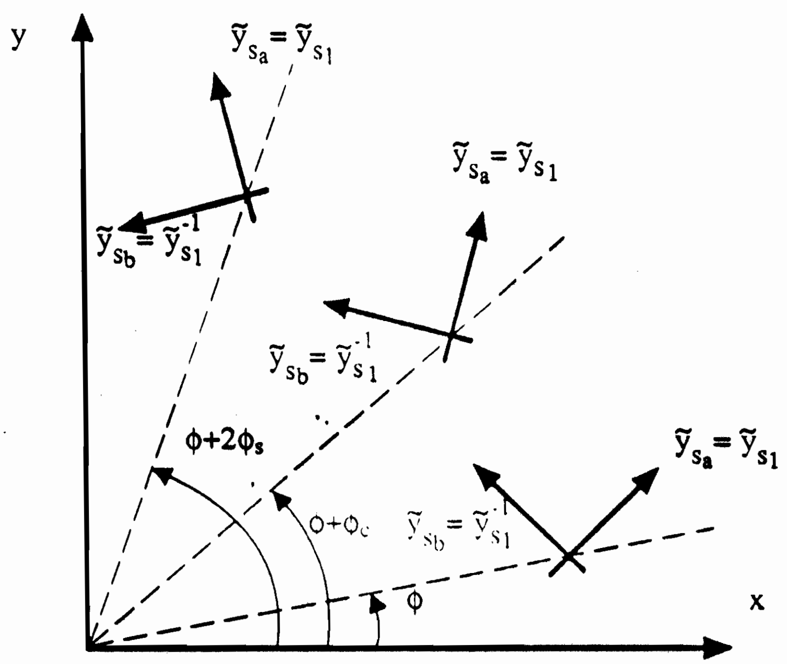
$$\tilde{y}_{s_a}(\Psi,\phi\,;s)\tilde{y}_{s_b}(\Psi,\phi\,;s) = 1 \tag{7.21}$$

so that the two eigenvalues are mutually inverse . A special case of this might have ∞ and 0 for $\tilde{y}_{s_a}$ and $\tilde{y}_{s_\infty}$ respectively ,corresponding to a uniconducting sheet with direction of conduction forming spiral paths on S , an example of C_∞ symmetry.

While C_{Nc} symmetry carries through for $\tilde{\mathbf{y}}_s$ it is not applicable in the same way to the fields and sources on S. As discussed in [6.2] the imposition of self-complementarity



A. Self complementarity



A. Self rotated inverse.

Figure 7.1 Dyadic Sheet Admittance

results in C_{2c} symmetry for sources and fields . This occurs because going to the complementary (dual) field is a $\pi/2$ rotation . For certain other cases of C_{Nc} symmetry , C_{2c} symmetry is a subgroup , specifically for

$$N=4M+2 = 2,6,10,... \qquad M=0,1,2.... \tag{7.22}$$

As an example a uniform plane wave with direction of incidence parallel to $\mathbf{1}_z$ induces C_{2c} sources on a C_{2c} admittance sheet. The fields resulting from C_{2c} sources have the special property that on the z axis (away from S) they are TEM waves [6.2], i.e. E and H are at right angles to each other.and the z axis (propagating outward) and related by Z, The polarization can vary with time or frequency (as in the case of spiral or scimitar like C_{2c} antennas). The TEM property is restricted in general to the z axis except in cases such as flat conical structures which support a TEM mode everywhere.

C_{2c} is an important special case . There is the usual self-complementary antenna consisting of conducting sheets in C_2 geometry excited by a single source at $\Psi = 0$. As indicated in Fig. 7.2 this can be generalized by a terminating impedance sheet of sheet resistance of Z/2 for all frequencies. If the resistive sheet is truncated (to give a finite size antenna)at radius $\Psi_1 \gg \Psi_0$ (approximate radius of the antenna conductors). This impedance will still be approximately Z/2 at low frequencies. Using the previous discussion various portions of the structure can be uniconducting sheets or whatever , subject to the C_{2c} constraint.

Figure 7.3 illustrates another notable result. Consider first an impedance sheet of value Z/2 in a C_{2c} structure. As shown in [6.2] , by performing integrals of **E** and **H** across such a region to give voltage and current , the ratio is just Z/2. Letting the region become electrically small one can consider the region as an impedance element provided there are alternate electric and magnetic boundaries in the C_{2c} sense. . Then connecting (at low frequencies) an impedance meter across the terminating pairs (conductors along the boundaries) gives Z/2 for the measurable impedance (neglecting stray inductance and capacitance). Scaling the result let the impedance sheet be of value $\tilde{Z}_s$; the impedance element is just $\tilde{Z}_s$. This applies to all shorts of shapes (circles ,stars, etc.) all of which have this simple answer. (This can be also shown from complex variable theory on interchange of electric and magnetic potentials.) More general C_{2c} admittances can be included in this region (involving slots,conductive strips ,etc) and scaled in the same way to give $\tilde{Z}_s$ for the overall impedance.

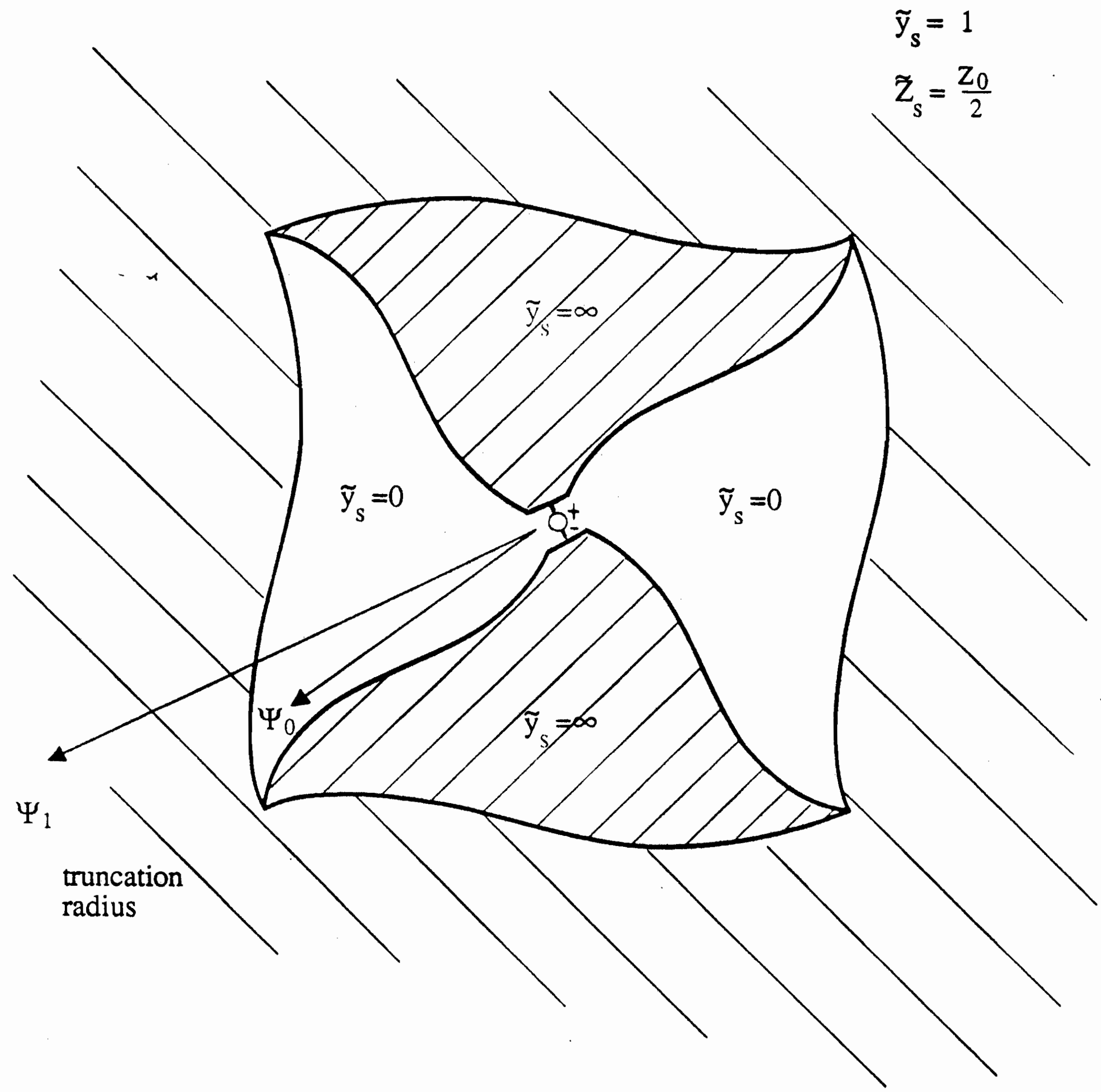


Figure 7.2 Self-Complementary Antenna with Exterior Resistive Termination

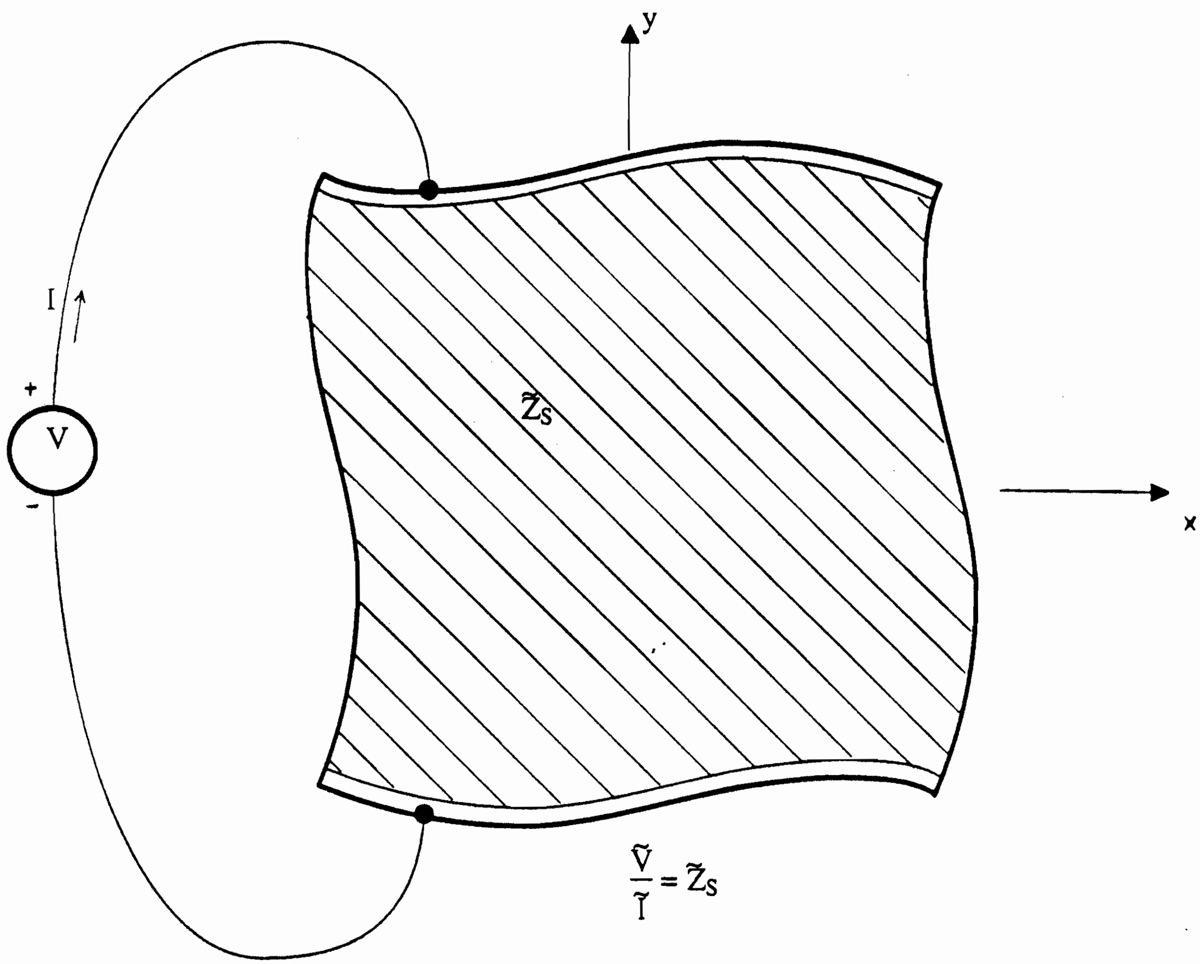


Figure 7.3 C_4 Impedance Sheet with Alternating Electric and Magnetic Boundaries.

VIII. SPACE GROUPS: TRANSLATION PLUS POINT SYMMETRY GROUPS IN THREE OR LESS DIMENSIONS

Consider the translation group T_d for d=1,2,3.. (dimension of the space) given by

$$T_d = \{ T_d(\xi) \mid \mathbf{r} \rightarrow \mathbf{r} + \xi \} \tag{8.1}$$

with **r** the position vector and ξ the arbitrary translation vector in d-dimensional space (real). This is a Lie group (continuous) with d parameters. For discrete translations we have ξ restricted as

$$\xi = \sum_{n=1}^{d} m_n \xi_n \quad , m_n \equiv \text{integers} \tag{8.2}$$

where the ξ_n are d basis vectors for the d-dimensional space. Starting from some origin (say $\mathbf{r} = \mathbf{0}$) then (8.2) generates a lattice by taking all integer values (+,- and 0) for the various m_n . This lattice is a periodic structure and moving from one lattice point to another gives the same local environment. It is known that such lattices can be constructed in 14 different ways for N=3 (Bravais lattices) [9.6]. Such lattices are very important to crystallography.

Consider rotations about any point (taken first as $\mathbf{r} = \mathbf{0}$)

$$\begin{aligned}
&O_2^+ = C_\infty \equiv \text{ rotations preserving } x^2+y^2 && (d=2) \\
&O_2 = C_{\infty a} \equiv \text{ rotations and reflections} && (d=2) \\
&O_3^+ \equiv \text{ rotations preserving } x^2+y^2+z^2 && (d=3) \\
&O_3 \equiv \text{ rotations and reflections} && (d=3)
\end{aligned} \tag{8.3}$$

The superscript + indicates only proper rotations ,referring to the +1 determinant of the associated rotation matrix representation. In other notations this superscript is replaced by a prefix S (special) . These proper rotation groups are Lie groups (continuous). Combining the above groups with translations gives the space groups E_d (Euclid) consisting of all transformations keeping distances invariant , or E_d^+ if only proper rotations are included with translations. Note that the group elements are not simple matrix multiplication but take the form

$$E_d = \{ (\Omega, \xi) \mid \mathbf{r} \rightarrow \Omega \cdot \mathbf{r} + \xi \} \tag{8.4}$$

$\Omega \equiv$ rotations and reflections in d-dimensional Euclidian space

($\Omega^+ \Rightarrow$ proper rotation)

$\xi \equiv$ translations in d-dimensional Euclidian space.

Note that one can generalize E_d to the Galilean group by including time as a parameter by a substitution of the form

$$\xi = \mathbf{v}\, t + \xi_0 \tag{8.5}$$

with the introduction of the constant velocity **v** with a time displacement (say t_0) as well [9.8].

Considering discrete translations (as in (8.2)) there are 73 types of space groups (designated as symmorphic) when O_3 is combined with discrete translation [9.6]. Furthermore one can combine translation with rotation and reflection by screw rotation axes and glide reflection planes ,respectively, giving another 157 types of non symmorphic space groups.

Electromagnetics has not made as much use of the space groups as has crystallography. However, in one dimension, this is commonly employed in a form known as Floquet's theorem. This is applied to periodically loaded waveguides such as slow-wave structures [9.3,9.4]. If translation is taken as the z direction with period z_0 then in terms of the electric field we have

$$\tilde{\mathbf{E}}(\mathbf{r},s) = e^{-\tilde{\gamma}_z(s)z}\, \tilde{\mathbf{E}}_p(\mathbf{r},s)$$

$$\tilde{\mathbf{E}}_p(\mathbf{r}+nz_0\mathbf{1}_z,s) = \tilde{\mathbf{E}}_p(\mathbf{r},s) \tag{8.6}$$

The one need only solve for the periodic function over one unit cell extending z_0 in the z direction together with periodic boundary condition. This leads to Bloch waves with pass and stop bands similar to those encountered in solid state physics these properties appearing in $\tilde{\gamma}_z(s)$. These results are of course generalizable to two and three dimensions using a periodicity as in (8.2).

Another common structure is the conducting helix, the basis for the helical antenna . This is a case of screw rotation symmetry where in going from z to z_0 there is a uniform rotation by $\pm 2\pi$ around the axis . This is a higher symmetry than merely periodic as in (8.6).

Items like diffraction gratings are another common periodic structure , which though practically are not infinite in extent , can usefully be treated as though they are infinite. While these are traditionally periodic on one direction ,they could be made periodic in two dimensions on a plane. Perhaps such structures (and even three-dimensional large arrays of antennas and/or scatterers) need to be investigated for the future design of electromagnetic devices.

IX. SIMILARITY GROUPS: ROTATION PLUS UNIFORM ISOTROPIC DILATION IN THREE OR LESS DIMENSIONS

The space groups E_d in the previous section are subgroups of the affine groups in d dimensions [9.19]

$$\Lambda_d = \left\{ (\mathbb{A},\xi) \middle| \mathbf{r} \rightarrow \mathbb{A}\cdot\mathbf{r}+\xi \right\}$$

$$\det(\mathbb{A}) \neq 0 \tag{9.1}$$

$\mathbb{A} \equiv$ rotation , stretching and shear in d-dimensional Euclidean Space

In the previous section $\mathbb{A}$ was restricted to rotations Ω (with determinant ± 1) . Here also we limit the discussion to real coordinates.

Another important subgroup is the linear group [9.8]

$$L_d = \left\{ \mathbb{A} \middle| \mathbf{r} \rightarrow \mathbb{A}\cdot\mathbf{r} \right\} \tag{9.2}$$

where now the coordinate origin is a special point that is invariant under this transformation. At this stage , besides rotation , $\mathbb{A}$ includes stretching (or compression) and shear , i.e. on transformation the shape of a geometric object is distorted.

In previous sections geometrical transformations were such as to bring the object into congruence with itself. Now the concept is similitude , i.e. after transformation the size (but not the shape) of the object is changed , this change in scale being a uniform dilation (expansion or shrinking), the dilated body being congruent with the original (including rotation and reflection). The electromagnetic significance concerns scaling with frequency or time. In the sense of wavelength λ ,dilation by a factor δ means that the object responds the same way at a wavelength $\delta\lambda$ (including appropriate rotation). This is an important concept in the design of what are termed frequency-independent antennas [8.4].

Keeping the shape of the object the same means that there be no shear in the transformation and that the stretching (or contraction) be the same in all directions (i.e. isotropic). Furthermore this must be the same for all points . Let us refer to this as a uniform dilation [9.23,9.24] . Including rotation and reflection (improper rotation) this leads to the group

$$\Delta_d = \left\{ \mathbb{\Delta} \middle| \mathbf{r} \rightarrow \mathbb{\Delta}\cdot\mathbf{r} \right\}$$

$$\mathbb{\Delta} = \delta\,\Omega \text{ , dilation factor} \equiv \delta > 0 \tag{9.3}$$

which might be referred to as the similarity group [9.20] or uniform dilation /rotation group. Note that for $\delta=1$ this reduces to O_d (section VI) which has also discrete subgroups. Now include the changes of scale (similarity) we can expect some of these rotation/reflection groups to play an important role. Noting that

$$\det(\mathbb{\Delta}) = \delta^d \det(\Omega)$$

$$\det(\Omega) = \pm 1 \tag{9.4}$$

one might think of this as a kind of complex rotation group , since n applications of the transformation expands the scale as δ^n , an exponential growth. Furthermore this implies that (for $\delta \neq 1$) the group is not closed , but applies to objects infinite in extent with a periodicity (unit cell) that expands proportional to r as one goes out from the origin. Such structures are referred to as log periodic,i.e. uniform decrements of $\ell n(r)$ for the unit cell of the structure. Including the rotation gives log spirals.

In two dimensions the rotation dyadic Ω corresponds to $(C_{n,m}(\phi))$ (plus a possible reflection) as in section VI. This being a rotation in a plane , such is appropriate for planar log periodic and spiral structures . As planar structures they can also be chosen from the set of self-complementary structures for additional symmetry and associated electromagnetic properties (section VII). Note that since there is dilation in radius as ϕ increases there need be no closure, i.e. rotation by 2π need not replicate the object at the new radius , i.e. the group element rotation need not be a submultiple of 2π except in the case where $\delta=1$.

In three dimensions the rotation dyadic $\mathbf{\Omega}^{+}$ (with +1 determinant) can be shown to be a three-real-parameter operation with two parameters determining the orientation of an axis (a point on the unit sphere) and the third parameter being rotation about this axis [9.8]. Successive application of $\mathbf{\Omega}^{+}$, say as $\mathbf{\Omega}^{+\ell}$ (integer ℓ) , is just successive rotation about the same axis by the same rotation angle. So our similarity group has a particular preferred axis which we choose without loss of generality to be the z axis. Let the rotation about this axis by $\mathbf{\Omega}$ be just ϕ_1 . The the rotation by $\mathbf{\Omega}^{+\ell}$ is just rotation about the same axis by $\ell\phi_1$ (which need not equal any multiple of 2π except for $\ell=0$). Recognizing this preferred axis then the rotation is two dimensional as described by $(C_{n,m}(\phi))$.

Considering infinitesimally small rotations and dilations one can define a one-parameter Lie group via

$$\begin{aligned}
&\mathbf{r} = \Delta(\alpha)\cdot\mathbf{r}_0\\
&\Delta(\alpha) = e^{\alpha\delta_s}\, e^{\alpha\mathbf{\Omega}_s} = e^{\alpha[\delta_s \mathbf{1}+\mathbf{\Omega}_s]}\\
&\Delta(0) = \mathbf{1}\\
&\delta = e^{\alpha\delta_s} \;,\; \mathbf{\Omega} = e^{\alpha\mathbf{\Omega}_s}\\
&\det(\mathbf{\Omega}) = e^{\alpha \mathrm{tr}(\mathbf{\Omega}_s)} = +1 \;, \mathrm{tr}(\mathbf{\Omega}_s)=0
\end{aligned} \tag{9.5}$$

where $\mathbf{r}_0$ is some staring coordinate of interest with spherical coordinates (r_0,θ_0,ϕ_0) . Note that $\mathbf{r}_0$ can be chosen as any spatial point on our object of interest , and in fact applies to every such point. The exponential nature of the transformation dyadic assures that successive dot multiplication by Δ gives

$$\Delta(\alpha_2)\cdot\Delta(\alpha_1) = \Delta(\alpha_1)\cdot\Delta(\alpha_2) = \Delta(\alpha_1+\alpha_2) \tag{9.6}$$

so that the group is also commutative.

Recalling that $\mathbf{\Omega}$ gives a preferred rotation axis which we take as the z axis , then the rotation affects the x and y coordinates (or just ϕ in cylindrical or spherical coordinates). Our rotation element can then be written (recall section VI)

$$\mathbf{\Omega} = \begin{pmatrix} \cos(\alpha\phi_s) & -\sin(\alpha\phi_s) & 0\\ \sin(\alpha\phi_s) & \cos(\alpha\phi_s) & 0\\ 0 & 0 & 1\end{pmatrix} = e^{\alpha\phi_s \begin{pmatrix} 0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}} \tag{9.7}$$

so that ϕ_s is some scale factor for the rotation just as δ_s is for dilation . Then we have

$$\Delta(\alpha) = e^{\alpha\left[\delta_s \mathbf{1}+\phi_s \begin{pmatrix} 0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\right]} = e^{\alpha\begin{pmatrix} \delta_s & -\phi_s & 0\\ \phi_s & \delta_s & 0\\ 0 & 0 & \delta_s\end{pmatrix}} \tag{9.8}$$

and we identify

$$\mathbf{\Omega}_s = \phi_s \begin{pmatrix} 0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \tag{9.9}$$

Interpreting this geometrically , beginning with some $\mathbf{r}_0$ this produces a conical spiral described by

$$(r,\theta,\phi) = \left(r_0\, e^{\alpha\delta_s},\ \theta_0, \phi_0+\alpha\phi_s\right) \qquad (9.10)$$

$$-\infty<\alpha<\infty$$

Essentially the shape of this spiral is governed only by the ratio $\frac{\delta_s}{\phi_s}$ i .e. the ratio of dilation to rotation. This is also referred to as an equiangular spiral since the spiral path makes a constant angle with the radius vector. Starting from some (θ_0,ϕ_0) produces one spiral ; another choice (with fixed r_0) produces another spiral and so on. Thinking of these as thin conductors (preferably with coniclay tapered cross sections) produces the usual conical spiral antennas [8.1,8.2,8.3,8.4]. Of course the conductors need not be thin , but should meet the spiral condition (9.10) for every point. (Truncation issues are another matter.) Note that projected on the unit sphere such spirals are circles described by a constant θ ($=\theta_0$) . Special cases are conical conductors (no variation in (θ,ϕ) , only dilation in r) defined by straight lines from the origin. With at least two such independent conducting cones, the structures can support TEM propagation in the radial direction[9.2].

Letting the transformation dyadic in (9.3) be discrete other possibilities emerge . Again Ω has an axis of rotation which we take as the z axis. Any starting $\mathbf{r}_0$ (or (r_0,θ_0,ϕ_0)) in spherical coordinates now generates a discrete type of logarithmic spiral. Projected on the unit sphere proper rotations have ϕ increased by some constant amount , say ϕ_1 , on each operation. Since θ is still constant (i.e. θ_0) these points all lie on a circle, but these points need not repeat (no closure) if $\frac{\phi_1}{2\pi}$ is irrational . Of course on each rotation there is also a dilation δ . If we choose $\theta=\frac{\pi}{2}$ for all points on the structures we have a planar log periodic structure (on the z=0 plane). Furthermore if there is no rotation (i.e. $\phi_1 = 0$) This applies to the simpler log periodic arrays[8.1,8.2,8.3,8.4].

One can also adjoin symmetry planes (reflection) as part of Ω . A transverse (to z) symmetry plane repeats the part for z>0 at z<0 as well (R_t symmetry). This plane can also be for rotation-reflection symmetry with points alternating on opposite sides of the z=0 plane. Note closure on 2π rotation (or $2m\pi$) is not required due to the dilation , so this is not S_N but is more general.

One or more axial symmetry planes (R_a symmetry) can also be adjoined . This places some constraints on the rotation angle ϕ_1 consistent with the number of such symmetry planes, since on successive operations the symmetry planes must transform into themselves or other such axial symmetry planes. Thus the rotation part must take the form of C_N (closure) in this case.

Thus far we have considered a single elementary dyadic $\delta\,\Omega$ which (with the identity) gives the group structure . This can be generalized to some set of $\delta_n\Omega_n$ for n=1,2...N. This then considers the possibility of more than one rotation axis (including opposite senses of rotation on the same axis. Thus we are led to include forms of the dihedral symmetry D_N and the tetrahedral , octahedral and icosahedral symmetries (T,O and Y) as in section VI. For example one might have a succession of coaxial cubical- and octahedral -like objects, alternating and exponentially expanding (dilation) for log periodic property.

It would then seem that not only can log-periodic spiral structures be cast into a group theoretical form , but the group structure can be expanded to consider other possibilities as well. Perhaps a more systematic investigation of similarity groups (and their adjunction with other symmetries such as reciprocity and duality) can lead to new types of electromagnetic devices.

Another possible application of similarity groups is to electromagnetic interaction with fractal objects [8.5]. Fractal objects are those which can be at least approximately described by fractal geometry . Structures with this geometry can be constructed as self similar. As a practical matter this self similarity extends over some smallest some largest scale. (Even practical log periodic antennas have such a scale limitation.) Fractals can be used to describe various irregular structures such as rough surfaces and turbulent media ,allowing some analysis of electromagnetic scattering properties.

X. SPECIAL RELATIVITY: MINKOWSKI SPACE,LORENZ GROUP, AND POINCARE GROUP

Special relativity came about from a symmetry observation of Einstein concerning the invariance of physical laws (such as the Maxwell equations) from one inertial reference frame to another. In electromagnetics this has the useful property that one can consider the fields in the reference frame of an antenna or scattterer and then transform the fields to the reference frame of a moving observer. This allows one to solve "moving media " problems in the rest frame and consequently simplify the problem.

Here our concern is not with the detailed calculations of moving-media problems such as Doppler shift ,etc. Rather our concern is symmetry or invariances in electromagnetic quantities under spaecial-relativistic transformation. With this in mind let us consider the four-vector/dyadic form of Maxwell equations , potentials , etc. and then incorporate duality. This will be followed by the Lorentz transformation as a rotation in Minkowski space and various resulting invariances. Note that here our concern is with special relativity and not with curved space-time as encountered in general relativity. As such in the forthcoming discussions we choose do not distinguish covariant and contravariant vectors and tensors [1.8,9.2,9.9,9.10].

A Minkowski space and 4-vectors and dyadics.

Let us represent four-dimensional Minkowski space via the coordinate or position 4-vector

$$\mathbf{r}_p \equiv (\mathbf{r},T_p) = (x,y,z,T_p)$$
$$= x\mathbf{1}_x+y\mathbf{1}_y+z\mathbf{1}_z+T_p\mathbf{1}_{T_p} \tag{10.1}$$
$$T_p \equiv jpct ,p=\pm 1$$

With x,y,z,t all real , then the fourth coordinate is imaginary . There are ,however, two choices that one can use ,depending on the choice for p. An alternate ,commonly used formulation involves the use of a metric tensor with a minus sign for the time coordinate or space coordinates. Our present formulation , on the other hand , will allow for the direct generalization of the commonly used dyadic analysis in electromagnetic theory. The del operator generalizes to

$$\Box_p=(\nabla, \frac{\partial}{\partial T_p}) = (\frac{\partial}{\partial x} , \frac{\partial}{\partial y} ,\frac{\partial}{\partial z} ,\frac{\partial}{\partial T_p})$$
$$= \mathbf{1}_x \frac{\partial}{\partial x} + \mathbf{1}_y\frac{\partial}{\partial y} + \mathbf{1}_z\frac{\partial}{\partial z} + \mathbf{1}_{T_p}\frac{\partial}{\partial T_p} \tag{10.2}$$

The generalized Laplacian is then

$$\Box_p^2 = \Box_p\cdot \Box_p= \nabla^2 + \frac{\partial^2}{\partial T_p^2} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z} + \frac{\partial^2}{\partial T_p^2}$$
$$= \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \tag{10.3}$$

which is conveniently independent of p. Further note that

$$\Box_p\cdot \mathbf{r}_p \equiv 4 \text{ (dimension of Minkowski space)} \tag{10.4}$$
$$\mathbf{r}_p\cdot \mathbf{r}_p = \mathbf{r}\cdot\mathbf{r} - c^2t^2 = |\mathbf{r}|^2 - c^2t^2 = [\, |\mathbf{r}| +ct\,]\, [|\mathbf{r}|-ct]$$

This last quantity is not the same as $|\mathbf{r}_p|^2$ since it can be negative.

B. Four-current density ,four-potential, and field tensor.

The current density takes the form

$$\mathbf{J}_p(\mathbf{r}_p) = (\mathbf{J}(\mathbf{r},t), jpc\rho(\mathbf{r},t)) \tag{10.5}$$

so that the equation of continuity becomes

$$\Box_p \cdot \mathbf{J}_p(\mathbf{r}_p) = 0 = \nabla \cdot \mathbf{J}(\mathbf{r},t) + \frac{\partial}{\partial t}\rho(\mathbf{r},t) \tag{10.6}$$

Similarly the vector potential takes the form

$$\mathbf{A}_p(\mathbf{r}_p) = \left(\mathbf{A}(\mathbf{r},t)\,,\, j\frac{p}{c}\,\Phi(\mathbf{r},t) \right) \tag{10.7}$$

so that the usual Lorentz guage condition takes the form

$$\Box_p \cdot \mathbf{A}_p(\mathbf{r}_p) = 0 = \nabla \cdot \mathbf{J}(\mathbf{r},t) + \frac{1}{c^2}\frac{\partial}{\partial t}\Phi(\mathbf{r},t) \tag{10.8}$$

with the wave equation

$$\Box_p^2\, \mathbf{A}_p(\mathbf{r}_p) = -\mu\, \mathbf{J}_p(\mathbf{r}_p) \tag{10.9}$$

From the usual solutions for outgoing waves

$$\tilde{\mathbf{A}}(\mathbf{r},s) = \mu \int_V \tilde{G}(\mathbf{r}-\mathbf{r}';s)\tilde{\mathbf{J}}(\mathbf{r}',s)dV'$$

$$\tilde{\Phi}(\mathbf{r},s) = \mu \int_V \tilde{G}_o(\mathbf{r}-\mathbf{r}';s)\rho(\mathbf{r}',s)dV' \tag{10.10}$$

$$\tilde{G}_o(\mathbf{r}-\mathbf{r}';s) = \frac{e^{-\gamma|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

we can take the inverse Laplace transform of the kernel (free space Greens function) as

$$G_o(\mathbf{r}-\mathbf{r}';t-t') \equiv G_o(\mathbf{r}_p-\mathbf{r}_p') = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\,\delta\left(t-t'-\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right) \tag{10.11}$$

This allows us to write

$$\mathbf{A}(\mathbf{r},t) = \mu \int_V \frac{\mathbf{J}(\mathbf{r}',t-\frac{|\mathbf{r}-\mathbf{r}'|}{c})}{4\pi|\mathbf{r}-\mathbf{r}'|}dV'$$

$$\Phi(\mathbf{r},t) = \frac{1}{\varepsilon} \int_V \frac{\rho(\mathbf{r}',t-\frac{|\mathbf{r}-\mathbf{r}'|}{c})}{4\pi|\mathbf{r}-\mathbf{r}'|}dV' \tag{10.12}$$

$$\mathbf{A}_p(\mathbf{r}_p) = \mu \int_V \frac{\mathbf{J}_p(\mathbf{r}_p'')}{4\pi|\mathbf{r}-\mathbf{r}'|}dV'$$

$$\mathbf{r}_p'' = (\mathbf{r}',\, jp[ct-|\mathbf{r}-\mathbf{r}'|])$$

Here we can see the difference between real space and real time through the lack of symmetry in the above formula which is an integral over all space , but only over retarded time for which $\mathbf{r}_p \cdot \mathbf{r}_p$ in (9.4) is negative . This can be thought of also in terms of the usual light cone using "time-like" coordinates ($\mathbf{r}_p \cdot \mathbf{r}_p$ negative). The 4-vector notation should then not lead one to obscure the different roles of $\mathbf{r}$ and t since T_p is imaginary.

The Maxwell equations can be written as

$$\mathbf{J}_p(\mathbf{r}_p) = \begin{pmatrix} \mathbf{J} \\ jpc\rho \end{pmatrix} = \begin{pmatrix} \nabla \times \mathbf{H} - j\frac{p}{Z}\frac{\partial \mathbf{E}}{\partial T_p} \\ j\frac{p}{Z}\nabla \cdot \mathbf{E} \end{pmatrix}$$

$$=\begin{pmatrix} \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} - j\frac{p}{Z}\frac{\partial E_x}{\partial T_p} \\ \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} - j\frac{p}{Z}\frac{\partial E_y}{\partial T_p} \\ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} - j\frac{p}{Z}\frac{\partial E_z}{\partial T_p} \\ j\frac{p}{Z}[\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}] \end{pmatrix} = j\frac{p}{Z}\,\Box_p\cdot\mathbb{F}_p(\mathbf{r}_p)$$

$$\mathbb{F}_p(\mathbf{r}_p) = \begin{pmatrix} 0 & jpZH_z & -jpZH_y & E_x \\ -jpZH_z & 0 & jpZH_x & E_y \\ jpZH_y & -jpZH_x & 0 & E_z \\ -E_x & -E_y & -E_z & 0 \end{pmatrix} \tag{10.13}$$

this last expression defining one form of the skew symmetric field tensor . Here we have not invoked the Lorentz force law to define this tensor , but have found it directly from the Maxwell equations.

In terms of the vector potential we have

$$\mathbf{E}(\mathbf{r},t) = -\nabla\Phi(\mathbf{r},t) - \frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t}$$

$$\mathbf{H}(\mathbf{r},t) = \frac{1}{\mu}\nabla\times\mathbf{A}(\mathbf{r},t) \tag{10.14}$$

$$\mathbb{F}(\mathbf{r}_p) = \begin{pmatrix} 0 & jpc[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}] & -jpc[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}] & -\frac{\partial \Phi}{\partial x} - jpc\frac{\partial A_x}{\partial T_p} \\ -jpc[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}] & 0 & jpc[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}] & -\frac{\partial \Phi}{\partial y} - jpc\frac{\partial A_y}{\partial T_p} \\ jpc[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}] & -jpc[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}] & 0 & -\frac{\partial \Phi}{\partial z} - jpc\frac{\partial A_z}{\partial T_p} \\ \frac{\partial \Phi}{\partial x} + jpc\frac{\partial A_x}{\partial T_p} & \frac{\partial \Phi}{\partial y} + jpc\frac{\partial A_y}{\partial T_p} & \frac{\partial \Phi}{\partial z} + jpc\frac{\partial A_x}{\partial T_p} & 0 \end{pmatrix}$$

$$= jpc\left[\Box_p \mathbf{A}_p(\mathbf{r}_p) - (\Box_p \mathbf{A}_p(\mathbf{r}_p))^T\right]$$

which is the usual result [9.9,9.10] generalized to the two posible choices for p.

C. Inclusion of duality to obtain combined four-dimensional quantities

Including the ficticious magnetic current density we have the combined current density and field as discussed in section III. Thus we have the combined 4-current density

$$\mathbf{J}_{p,q}(\mathbf{r}) \equiv (\mathbf{J}_q(\mathbf{r},t), pjc\rho_q(\mathbf{r},t))$$

$$\nabla\cdot\mathbf{J}_q(\mathbf{r},t) = -\frac{\partial}{\partial t}\rho_q(\mathbf{r},t) \tag{10.15}$$

$$\rho_q(\mathbf{r},t) = \rho(\mathbf{r},t) + j\frac{q}{Z}\rho_m(\mathbf{r},t) = \varepsilon\,\nabla\cdot\mathbf{E}_q(\mathbf{r},t)$$

$$\Box_p\cdot\mathbf{J}_{p,q}(\mathbf{r}) = 0$$

Note that this holds for all combinations of p and q (independently +1 and -1). From combined vector and scalar potentials

$$\mathbf{A}_q(\mathbf{r},t) = \mathbf{A}(\mathbf{r},t) + jqZ\mathbf{A}_m(\mathbf{r},t)$$

$$\Phi_q(\mathbf{r},t) = \Phi(\mathbf{r},t) + jqZ\Phi_m(\mathbf{r},t)$$

$$\mathbf{E}(\mathbf{r},t) = -\nabla\Phi(\mathbf{r},t) - \frac{\partial}{\partial t}\mathbf{A}(\mathbf{r},t) - \frac{1}{\varepsilon}\nabla\times\mathbf{A}_m(\mathbf{r},t) \tag{10.16}$$

$$\mathbf{H}(\mathbf{r},t) = \frac{1}{\mu}\nabla\times\mathbf{A}(\mathbf{r},t) - \nabla\cdot\Phi_m(\mathbf{r},t) - \frac{\partial}{\partial t}\mathbf{A}_m(\mathbf{r},t)$$

$$\mathbf{E}_q(\mathbf{r},t) = -\nabla\Phi_q(\mathbf{r},t) - \frac{\partial}{\partial t}\mathbf{A}_q(\mathbf{r},t) + jqc\nabla\times\mathbf{A}_q(\mathbf{r},t)$$

the combined 4-potential is

$$\mathbf{A}_{p,q}(\mathbf{r}_p) = (\mathbf{A}_q(\mathbf{r},t), j\frac{p}{c}\Phi_q(\mathbf{r},t))$$

$$\Box_p\cdot\mathbf{A}_{p,q}(\mathbf{r}) = 0 = \nabla\cdot\mathbf{A}_q(\mathbf{r},t) + \frac{1}{c^2}\frac{\partial}{\partial t}\Phi_q(\mathbf{r},t) \tag{10.17}$$

$$\Box_p^2\mathbf{A}_{p,q}(\mathbf{r}) = \mu\,\mathbf{J}_{p,q}(\mathbf{r}_p)$$

In terms of the combined current density (10.12) becomes

$$\mathbf{A}_q(\mathbf{r},t) = \mu\int_V \frac{\mathbf{J}_q(\mathbf{r}',t-\frac{|\mathbf{r}-\mathbf{r}'|}{c})}{4\pi|\mathbf{r}-\mathbf{r}'|}dV'$$

$$\Phi_q(\mathbf{r},t) = \frac{1}{\varepsilon}\int_V \frac{\rho_q(\mathbf{r}',t-\frac{|\mathbf{r}-\mathbf{r}'|}{c})}{4\pi|\mathbf{r}-\mathbf{r}'|}dV'$$

$$\mathbf{A}_p(\mathbf{r}_p) = \mu\int_V \frac{\mathbf{J}_{p,q}(\mathbf{r}''_p)}{4\pi|\mathbf{r}-\mathbf{r}'|}dV' \tag{10.18}$$

$$\mathbf{r}''_{p,q} \equiv (\mathbf{r}', jp[ct-|\mathbf{r}-\mathbf{r}'|])$$

The combined Maxwell equations can now be written as

$$jqZ\mathbf{J}_{p,q}(\mathbf{r}_p) = jqZ\begin{pmatrix}\mathbf{J}_q\\ jpc\rho_q\end{pmatrix} = \begin{pmatrix}[\nabla\times + pq\frac{\partial}{\partial T_p}]\mathbf{E}_q\\ -pq\nabla\cdot\mathbf{E}_q\end{pmatrix}$$

$$= \Box_p\cdot\mathbb{E}_{p,q}(\mathbf{r}_p) \tag{10.19}$$

$$\mathbb{E}_{p,q}(\mathbf{r}_p) \equiv \begin{pmatrix} 0 & -E_{zq} & E_{yq} & -pqE_{xq} \\ E_{zq} & 0 & -E_{xq} & -pqE_{yq} \\ -E_{yq} & E_{xq} & 0 & -pqE_{zq} \\ pqE_{xq} & pqE_{yq} & pqE_{zq} & 0 \end{pmatrix}$$

$= -\mathbb{E}^T_{p,q}(\mathbf{r}_p)$ (skew symmetric)

This gives the combined field tensor. Note that p and q enter in a form which gives all real coefficients to the combined field componenets (three instead of six) . Choosing p=+1 and -1 gives two forms of this tensor . Then choosing q=1 and -1 and taking sum and difference to separate electric and magnetic terms gives a total of <u>four </u>separate forms af the field tensor .Note that

$$\Box_p \cdot (\Box_p \cdot \mathbb{E}_{p,q}(\mathbf{r}_p)) = qjZ \, \Box_p \cdot \mathbf{J}_{p,q}(\mathbf{r}_p) = 0 \tag{10.20}$$

Now introduce the concept of a dual of a 4x4 skew symmetric dyadic [1.8,9.9,9.10] . This done via a four index array or fourth rank tensor ($d_{n,m,n',m'}$) where

$$d_{n,m,n',m'} = \begin{cases} \frac{1}{2} \text{ for any even permutation of } (1,2,3,4) \\ -\frac{1}{2} \text{ for any odd permutation of } (1,2,3,4) \\ 0 \text{ otherwise (any two indices equal)} \end{cases} \tag{10.21}$$

Then an antisymetric dyadic (tensor) has the general form

$$\mathbb{X} = \begin{pmatrix} 0 & X_{1,2} & X_{1,3} & X_{1,4} \\ -X_{1,2} & 0 & X_{2,3} & X_{2,4} \\ -X_{1,3.} & -X_{2,3} & 0 & X_{3,4} \\ -X_{1,4} & -X_{2,4} & -X_{3,4} & 0 \end{pmatrix} = -\mathbb{X}^T \tag{10.22}$$

Its dual is defined by

$$\text{dual}(\mathbb{X}) \equiv \left(\sum_{n',m'} d_{n,m,n',m'} X_{n',m'} \right) \tag{10.23}$$

and explicitely given by

$$\text{dual}(\mathbb{X}) = \begin{pmatrix} 0 & X_{3,4} & -X_{2,4} & X_{2,3} \\ -X_{3,4} & 0 & X_{1,4} & -X_{1,3} \\ X_{2,4.} & -X_{1,4} & 0 & X_{1,2} \\ -X_{2,3} & X_{1,3} & -X_{1,2} & 0 \end{pmatrix}$$

As one would expect

$$\text{dual}(\text{dual}(\mathbb{X})) = \mathbb{X} \tag{10.25}$$

so that twice application of dual (...) produces the original skew symmetrical tensor.

Now go back to the combined-field tensor in (10.19) and observe

$$\text{dual}(\mathbb{E}_{p,q}(\mathbf{r}_p)) = pq \, \mathbb{E}_{p,q}(\mathbf{r}_p) \tag{10.26}$$

How symmetric! The skew symmetric combined-field tensor is its own dual or antidual depending on the sign of pq (=±1).

Relating the combined fields to the combined potentials one has from (10.16) the three independent combined-field components as components of

$$\mathbf{E}_q(\mathbf{r},t) = jpq\left[\nabla(j\frac{p}{c}\Phi_q(\mathbf{r},t)) - \frac{\partial}{\partial T_p}\mathbf{A}_q(\mathbf{r},t)\right] + jqc\nabla \times \mathbf{A}_q(\mathbf{r},t) \tag{10.27}$$

Divide the combined-field tensor into two parts as

$$\mathbb{E}_{p,q}(\mathbf{r}_p) = jqc \, \mathbb{X}^{(1)} + jpc \, \mathbb{X}^{(2)} \tag{10.28}$$

where

$$\mathbb{X}^{(1)} = \begin{pmatrix} 0 & \frac{\partial A_{xq}}{\partial y} - \frac{\partial A_{yq}}{\partial x} & \frac{\partial A_{xq}}{\partial z} - \frac{\partial A_{zq}}{\partial x} & -\frac{\partial}{\partial x}(\frac{jp}{c}\Phi_q)+\frac{\partial A_{xq}}{\partial T_p} \\ \frac{\partial A_{yq}}{\partial x} - \frac{\partial A_{xq}}{\partial y} & 0 & \frac{\partial A_{yq}}{\partial z} - \frac{\partial A_{zq}}{\partial y} & -\frac{\partial}{\partial y}(\frac{jp}{c}\Phi_q)+\frac{\partial A_{yq}}{\partial T_p} \\ \frac{\partial A_{zq}}{\partial x} - \frac{\partial A_{xq}}{\partial z} & \frac{\partial A_{zq}}{\partial y} - \frac{\partial A_{yq}}{\partial z} & 0 & -\frac{\partial}{\partial z}(\frac{jp}{c}\Phi_q)+\frac{\partial A_{zq}}{\partial T_p} \\ \frac{\partial}{\partial x}(\frac{jp}{c}\Phi_q)-\frac{\partial A_{xq}}{\partial T_p} & \frac{\partial}{\partial y}(\frac{jp}{c}\Phi_q)-\frac{\partial A_{yq}}{\partial T_p} & \frac{\partial}{\partial z}(\frac{jp}{c}\Phi_q)-\frac{\partial A_{zq}}{\partial T_p} & 0 \end{pmatrix}$$

$$= - \square_p \mathbf{A}_{p,q}(\mathbf{r}_p) + (\square_p \mathbf{A}_{p,q}(\mathbf{r}_p))^T \tag{10.29}$$

and

$$\mathbb{X}^{(2)} = \begin{pmatrix} 0 & -\frac{\partial}{\partial z}(\frac{jp}{c}\Phi_q) + \frac{\partial A_{zq}}{\partial T_p} & \frac{\partial}{\partial y}(\frac{jp}{c}\Phi_q) - \frac{\partial A_{yq}}{\partial T_p} & \frac{\partial A_{yq}}{\partial z} - \frac{\partial A_{zq}}{\partial y} \\ \frac{\partial}{\partial z}(\frac{jp}{c}\Phi_q) - \frac{\partial A_{zq}}{\partial T_p} & 0 & -\frac{\partial}{\partial x}(\frac{jp}{c}\Phi_q)+\frac{\partial A_{xq}}{\partial T_p} & \frac{\partial A_{zq}}{\partial x} - \frac{\partial A_{xq}}{\partial z} \\ -\frac{\partial}{\partial y}(\frac{jp}{c}\Phi_q) + \frac{\partial A_{yq}}{\partial T_p} & \frac{\partial}{\partial x}(\frac{jp}{c}\Phi_q) - \frac{\partial A_{xq}}{\partial T_p} & 0 & \frac{\partial A_{xq}}{\partial y} - \frac{\partial A_{yq}}{\partial x} \\ \frac{\partial A_{zq}}{\partial y} - \frac{\partial A_{yq}}{\partial z} & \frac{\partial A_{xq}}{\partial z} - \frac{\partial A_{zq}}{\partial x} & \frac{\partial A_{yq}}{\partial x} - \frac{\partial A_{xq}}{\partial y} & 0 \end{pmatrix}$$

$$\mathrm{dual}(\mathbb{X}^{(1)}) = \mathrm{dual}\left(- \square_p \mathbf{A}_{p,q}(\mathbf{r}_p)+(\square_p \mathbf{A}_{p,q}(\mathbf{r}_p))^T\right) \tag{10.30}$$

As an aid in this decomposition note that the pq coefficients (in fourth row and fourth column of the combined field-tensor of (10.19)) interchange the coefficients p and q in front of the two terms of (10.27) . Combining the results

$$\mathbb{E}_{p,q}(\mathbf{r}_p) = -jc\left[q(\square_p \mathbf{A}_{p,q}(\mathbf{r}_p)-(\square_p \mathbf{A}_{p,q}(\mathbf{r}_p))^T)+p\ \mathrm{dual}(\square_p \mathbf{A}_{p,q}(\mathbf{r}_p)-(\square_p \mathbf{A}_{p,q}(\mathbf{r}_p))^T)\right] \tag{10.31}$$

which evidently is consistent with (10.26) (self dual or antidual). Thus the combined-field tensor can be derived from the combined 4-potential in a way which generalizes the usual results (as in (10.14)).

D. Time reversal and magnetic current and charge.

Reversing the sign of p can be thought of as reversing time. In four dimensional space-time this a reflection, or improper rotation. In a deeper sense,however, one does not have time-reversal symmetry. While the differential equations have such symmetry the resulting fields do not. This is because the differential equations do not have a unique solution. When causality is imposed (fields coming after currents, or equivalently outgoing waves) the solution becomes unique and takes the form in terms of integrals over the currents with an outward going (retarded) Green's function . As in (10.12) and (10.18) note that the integration is over space (three dimensions) and not time , this variable being combined with spatial coordinates in the form of retarded time. So in in general there is not time-reversal symmetry , although there are special cases such as lossless cavities (no exterior radiation) when there is such symmetry (at least to a good approximation).

Reversing the sign of q can be thought of as reversing the magnetic field and magnetic current and charge. Upon reversal of spatial coordinates ($\mathbf{r} \rightarrow -\mathbf{r}$ or inversion) and conserving sign on the electric parameters, then there is a sign reversal on the magnetic parameters, thereby associating q with a spatial inversion , an improper rotation.

Alternatively one can reverse time in the Maxwell equations ($t \rightarrow -t$ or $p \rightarrow -p$), another improper rotation in four-dimensional space-time, again requiring reversal of the magnetic field ($q \rightarrow -q$). In this sense we note the conservation of the sign of pq in (10.19) and (10.26) in the combined-field dyadic. Of course, as discussed before, this property applies to the differential equations prior to imposition of causality.

So causality (initial conditions) breaks time-reversal symmetry. What then about magnetic-reversal symmetry? Again consider what happens under spatial inversion. If we insist on conserving electric charge (no sign reversal), on spatial inversion then there is a sign reversal on magnetic parameters including any magnetic charge. One could have adopted a convention of conserving magnetic charge on inversion, at the price of reversing electric charge. However, one cannot conserve both in this sense. If one is to have spatial inversion symmetry then one can choose which kind of charge to have and then define this as an electric charge (merely a question of labelling[9.9]). The magnetic field does not present the same difficulty since **ExH** remains outward on both reversal of coordinates and **H**, and **JxB** terms are conserved by reversal in both **B** and the cross product.

E. Special-relativity groups

Similar to the affine group in (9.1) we have the Poincare group

$$P = \{ (\mathbb{L}_p, \xi_p) \mid \mathbf{r}_p \rightarrow \mathbb{L}_p \cdot \mathbf{r}_p + \xi_p \} \tag{10.32}$$

where the translation in Minkowski space is

$$\xi_p = (\xi_x, \xi_y, \xi_z, \xi_{T_p}) , \ \xi_{T_p} \text{ in imaginary meters} \tag{10.33}$$

A subgroup of this is the Lorenz group

$$L_p = \{ \mathbb{L}_p \mid \mathbf{r}_p \rightarrow \mathbb{L}_p \cdot \mathbf{r}_p \} \tag{10.34}$$

The important term here is the Lorenz dyadic which can be thought as a complex rotation in Minkowski space. There are various forms this can take as one can find in the extensive literature (e.g [9.9,9.10,9.13,9.15,9.22]).

For present purposes consider two reference (inertial) frames, the second moving at a velocity **v** (real and constant) with respect to the first. Then defining

$$v \equiv |\mathbf{v}| \ , \ \boldsymbol{\beta} \equiv \frac{1}{c}\mathbf{v} \ , \ \beta \equiv \frac{v}{c} \ , \ \psi_p \equiv -jp\psi$$
$$\beta(1-\beta^2)^{-1/2} \equiv \sinh(\psi) = jp\sin(\psi_p) \tag{10.35}$$
$$(1-\beta^2)^{-1/2} = \cosh(\psi) = \cos(\psi_p)$$

we have the Lorenz transformation

$$\mathbf{r}_p^{(2)} = \mathbb{L}_p \cdot \mathbf{r}_p^{(1)} \tag{10.36}$$

The Lorenz dyadic can be constructed in two steps [9.22] as

$$\mathbb{L}^{(2)} \cdot \mathbf{r}_p^{(2)} = \mathbb{L}_p^{(z)} \cdot \mathbb{L}^{(1)} \cdot \mathbf{r}_p^{(1)} \tag{10.37}$$

The first step is a simple rotation of the spatial coordinates as

$$\mathbb{L}^{(n)} = \begin{pmatrix} & & & 0 \\ & \boldsymbol{\Omega}^{(n)} & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{10.38}$$

where $\boldsymbol{\Omega}^{(n)}$ is a rotation dyad (in O_3^+ as in section VII) making the spatial coordinates in $\mathbf{r}_p^{(2)}$ parallel to and in the same direction as the spatial coordinates in $\mathbf{r}_p^{(1)}$, after first rotating the

$z^{(2)}$ and $z^{(1)}$ axes parallel to and in the direction of **v** (a real velocity vector). The remaining space/time transformation is handled via

$$\mathbb{L}_p^{(z)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\psi_p) & -\sin(\psi_p) \\ 0 & 0 & \sin(\psi_p) & \cos(\psi_p) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh(\psi) & jp\sinh(\psi) \\ 0 & 0 & -jp\sinh(\psi) & \cosh(\psi) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (1-\beta^2)^{-1/2} & jp\beta(1-\beta^2)^{-1/2} \\ 0 & 0 & -jp\beta(1-\beta^2)^{-1/2} & (1-\beta^2)^{-1/2} \end{pmatrix} \tag{10.39}$$

$$\mathbb{L}_p^{(z)\,-1} = \mathbb{L}_{-p}^{(z)} = \mathbb{L}_p^{(z)\,T}$$

$$\det(\mathbb{L}_p^{(z)}) = +1$$

which is a form of complex proper rotation.

Assuming the spatial coordinates are already rotated to be parallel as above then $\mathbb{L}^{(1)} = \mathbb{L}^{(2)} = =(1_{n,m})$

$$\begin{pmatrix} x^{(2)} \\ y^{(2)} \\ z^{(2)} \\ T_p^{(2)} \end{pmatrix} = \mathbf{r}_p^{(2)} = \mathbb{L}_p^{(2)} \cdot \mathbf{r}_p^{(1)} = \begin{pmatrix} x^{(1)} \\ y^{(1)} \\ \cos(\psi_p)z^{(1)}-\sin(\psi_p) \\ \sin(\psi_p)z^{(1)}+\cos(\psi_p)T_p^{(1)} \end{pmatrix}$$

$$= \begin{pmatrix} x^{(1)} \\ y^{(1)} \\ \cosh(\psi)z^{(1)}+jp\sinh(\psi)T_p^{(1)} \\ -jp\sinh(\psi)z^{(1)}+\cosh(\psi)T_p^{(1)} \end{pmatrix} \tag{10.40}$$

In terms of real time t this is

$$\begin{pmatrix} x^{(2)} \\ y^{(2)} \\ z^{(2)} \\ jpct \end{pmatrix} = \begin{pmatrix} x^{(1)} \\ y^{(1)} \\ \cosh(\psi)z^{(1)}-\sinh(\psi)ct^{(1)} \\ jp[-\sinh(\psi)z^{(1)}+\cosh(\psi)ct^{(1)}] \end{pmatrix} = \begin{pmatrix} x^{(1)} \\ y^{(1)} \\ (1-\beta^2)^{-1/2}[z^{(1)}-\beta ct^{(1)}] \\ jp(1-\beta^2)^{-1/2}[ct^{(1)}-\beta z^{(1)} \end{pmatrix} \tag{10.41}$$

Since the Lorenz dyadic is just a proper rotation the other 4-vectors/dyadics transform as

$$\mathbf{J}_{p,q}^{(2)}(\mathbf{r}_p^{(2)}) = \mathbb{L}_p \cdot \mathbf{J}_{p,q}^{(1)}(\mathbf{r}_p^{(1)})$$

$$\mathbf{A}_{p,q}^{(2)}(\mathbf{r}_p^{(2)}) = \mathbb{L}_p \cdot \mathbf{A}_{p,q}^{(1)}(\mathbf{r}_p^{(1)}) \tag{10.42}$$

$$\mathbb{E}_{p,q}^{(2)}(\mathbf{r}_p^{(2)}) = \mathbb{L}_p \cdot \mathbb{E}_{p,q}^{(1)}(\mathbf{r}_p^{(1)}) \cdot \mathbb{L}_p^{-1}$$

where $\mathbb{L}_p$ can be replaced by $\mathbb{L}_p^{(z)}$ for a simpler version if the two sets of coordinates are aligned as discussed previously.

F.Invariances.

Our concern here is symmetry , one aspect of which is the invariance of the forms of the various equations (and solutions) with respect to the reference frame (Lorenz invariance). This can be thought of in the usual 3-vector/dyadic equations or in the 4-vector/dyadic equations , both of which express duality by combining electric and magnetic parameters together (and thereby simplifying many of the equations).

In addition to the differential equations (and the solution) there are various conservation laws such as energy and reciprocity (section IV) which apply in all inertial frames. In 4-vector/dyadic form there are related concepts concerning the energy momentum tensor . A remarkable property of the combined field 4-dyadic

$$\mathbb{E}_{p,q}(\mathbf{r}_p)\cdot \mathbb{E}_{p,q}(\mathbf{r}_p) = \mathbf{E}_q(\mathbf{r},t)\cdot\mathbf{E}_q(\mathbf{r},t)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{10.43}$$

which can be verified by multiplying out the elements in (10.19) . Rotating the above 4-dyadic to transform from $\mathbf{r}_p^{(1)}$ to $\mathbf{r}_p^{(2)}$ gives

$$\mathbf{E}_q^{(2)}\cdot\mathbf{E}_q^{(2)} = \mathbf{E}_q^{(1)}\cdot\mathbf{E}_q^{(1)} \tag{10.44}$$

This says that the dot product of the combined field with itself (loosely the "square") is invariant to Lorenz transformation as in the square of the combined-field 4-dyadic. Separating the electric and magnetic fields gives the usual results [9.10]

$$\begin{gathered} \mathbf{E}^{(2)}\cdot\mathbf{E}^{(2)} - Z^2\mathbf{H}^{(2)}\cdot\mathbf{H}^{(2)} = \mathbf{E}^{(1)}\cdot\mathbf{E}^{(1)} - Z^2\mathbf{H}^{(1)}\cdot\mathbf{H}^{(1)} \\ \mathbf{E}^{(2)}\cdot\mathbf{H}^{(2)} = \mathbf{E}^{(1)}\cdot\mathbf{H}^{(1)} \end{gathered} \tag{10.45}$$

again showing the compactness of the combined-field notation, and the symmetry inherent in it. Further consideration of the 4-dyadic gives

$$\begin{gathered} \det(\mathbb{E}_{p,q}(\mathbf{r}_p)) = [\mathbf{E}_q(\mathbf{r},t)\cdot\mathbf{E}_q(\mathbf{r},t)]^2 \\ \mathrm{tr}(\mathbb{E}_{p,q}(\mathbf{r}_p)) = 0 \end{gathered} \tag{10.46}$$

which are also Lorenz invariant . Noting from (10.43) that the eigenvalues of the square of the 4-dyadic are all $\mathbf{E}_q\cdot\mathbf{E}_q$, then the four eigenvalues of $\mathbb{E}_{p,q}$ are just

$$\lambda_q(\mathbb{E}_{p,q}(\mathbf{r}_p)) = \pm[\mathbf{E}_q(\mathbf{r},t)\cdot\mathbf{E}_q(\mathbf{r},t)]^{1/2} \tag{10.47}$$

with each sign taken exactly twice . These eigenvalues are of course invariant to Lorenz transformations while the eigenvectors are of course changed (rotated) by multiplication by $\mathbb{L}_p$. Note that for a TEM plane wave , these eigenvalues are all zero , and the square of the 4-dyadic combined in (10.43) is a zero dyadic.

In 4-vector/dyadic form ,there are conservation concepts concerning the energy/momentum tensor . In this paper our concern has been with electromagnetic sources and fields , and not with associated mechanical concepts of force and momentum.

G. Symmetry in space/time

With the Lorenz transformation one can consider symmetries in 4-dimensional space/time. One can have a group of general "rotational" form

$$G_{L_p} = \{\mathbb{G}_{L_p}^{(n)} \mid n=1,2,\ldots N\} \tag{10.48}$$

where N can be finite or infinite . One of these elements (say the Nth) is just the identity

$$\mathbb{G}_{Lp}^{(N)} = (1_{n,m}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{10.49}$$

The Lorenz transformation to some other inertial frame , say $\mathbb{G}_{Lp}^{(1)}$ has the inverse $(\psi \rightarrow -\psi)$ as $\mathbb{G}_{Lp}^{(2)}$, forming a three element group . The set of all proper Lorenz transformations forms a Lie group (infinite number of elements).

Let us suggest some generalization of some of the 2-and 3-dimensional symmetries discussed previously. For convenience use $\mathbb{L}_n$ to rotate the spatial coordinates in the n^{th} reference frame to align the z axis with the velocity vector $\mathbf{v}$. Then one might consider an object with C_N symmetry with respect to an N-fold axis parallel to $\mathbf{v}$. To an observer on the z axis but in a different reference frame (say zero velocity against which $\mathbf{v}$ is defined) the object still has C_N symmetry even though the Lorenz contraction changes the size of the object in its z coordinates. Note that even if the body is in constant rotation about the z axis the C_N symmetry is still maintained in the observers reference frame.Points on the object then proceed to form helices as time progresses . The world line of each point on the object is a helix, the C_N object forming a set of world lines that generate a body with both rotational (C_N) and screw-translation symmetry.

Considered in a general sense one has a symmetric object when the 4-dimensional representation of an object (world line set) transforms into itself under a Lorenz transformation (or more generally a Poincare transformation). To form a symmetry group the set of these transformations need only form a group.

A commonly used kind of symmetry involves transforming the reference frame to some kind of "average" frame for the set of objects under consideration. For identical particles and/or antiparticles this is often referred to as center-of-mass coordinates . Our concern is not with mass but shapes , positions , orientations and velocities of a set of electromagnetic objects, i.e. antennas and scatterers . Nevertheless, optimal choice of reference frame can help the engineering problem by bringing out symmetry properties.

Here we have just scratched the surface of the various space/time symmetries that one may find useful for electromagnetic design. Much more work would seem to be needed in this area.

XI. CONCLUDING REMARKS

Well , this has been some odyssey! Beginning with the Maxwell equations we have wandered through the symmetries inherent in the equations (duality,reciprocity). Next geometrical symmetries (rotation, etc.) were explored and combined with duality (self complementarity). Then going to space/time further symmetries were found in the Maxwell equations and "geometrical" symmetries in four dimensions discussed . Even with the large amount of material included here one can think of this as an outline of the subject. Much more can be done to fill the various parts . Hopefully this has helped define the questions and pointed to new areas of research.

How far might one extend the concept of symmetry in electromagnetics? A recent book [9.2] has shown how to extend our usual kinds of electromagnetic fields in euclidean space (the formal fields) to other useful cases of inhomogeneous and/or anisotropic media (ideal lenses). These lenses can themeselves have various symmetries. In the formal coordinates we can have various symmetries in the fields, antennas and scatteres . When transformed in a differential geometric sense to the real coordinates (with the lens medium) these symmetries may not be preserved in a purely geometric sense, but they are still there in the equations. So general symmetry concepts can apply to objects in such media.

Where else might symmetry be found in electromagnetics? Perhaps you the reader will help.

Appendix A : Properties of Finite Groups

Continuing the discussion of groups from section II, let our group $\mathcal{G}$ have a finite number of elements ℓ_0 ,the order of the group. As a group we already have the group operation (symbolic multiplication , not necessarily commutative), associativity , the identity element , and inverse element, and inverse defined. The following is a summary of the group properties. More detail can be found in various texts (e.g. [9.5,9.6,9.8]).

Isomorphism

Two groups are isomorphic iff their elements can be put into one-to-one correspondence including the combinations by the group operations. An example of this is the multiplicity of matrix representations of a group (as in section II).

Subgroup

A subgroup $\mathcal{G}_s$ of a group $\mathcal{G}$ is itself a group of order ℓ_s. All elements of $\mathcal{G}_s$ are also elements of $\mathcal{G}$. Each subgroup has an index.

$$\text{index of } \mathcal{G}_s \equiv \frac{\ell_0}{\ell_s} = \text{an integer} \tag{A.1}$$

A proper subgroup has $\ell_s \neq 1, \ell_0$ and equivalently index $\neq \ell_0, 1$ i.e. is not the group $\{(1)\}$ or $\mathcal{G}$.

Cosets

For any subgroups $\mathcal{G}_s$ of $\mathcal{G}$ with any element $(\mathcal{G})_\ell$ form the sets (complexes)

$$(\mathcal{G})_\ell \, \mathcal{G}_s \equiv \text{left cosets}$$
$$\mathcal{G}_s (\mathcal{G})_\ell \equiv \text{right cosets} \tag{A.2}$$

Commutative Group

A commutative (or Abelian) group has the property

$$(\mathcal{G})_\ell \, (\mathcal{G})_{\ell'} = (\mathcal{G})_{\ell'} \, (\mathcal{G})_\ell \tag{A.3}$$

for all ℓ and ℓ' ranging over the group elements.

Cyclic group

A cyclic group has elements defined by

$$(\mathcal{G})_\ell \equiv (\mathcal{G})_1^\ell \,, (\mathcal{G})_{\ell_0} \equiv (1) \text{ identity.} \tag{A.4}$$

All cyclic groups of the same order ℓ_0 are isomorphic to each other and have the same irreducible representation

$$(\mathcal{G})_\ell \rightarrow e^{j2\pi\frac{\ell}{\ell_0}} \text{ (a scalar or dimension d=1)} \tag{A.5}$$

This is a commutative group. All subgroups of a cyclic group are themselves cyclic.

Groups of prime order

If ℓ_0 is a prime number (cannot be factored into a product of integers excluding 1 and ℓ_0) , then $\mathcal{G}$ has no proper subgroups. Furthermore $\mathcal{G}$ is cyclic.

Symmetric group

This is the group of permutations of m symbols and has order m! . It is said to have degree m.

Cayley's theorem

Every group $\mathcal{G}$ of order ℓ_0 is isomorphic to a subgroup of the symmetric group of permutations of ℓ_0 symbols (order $\ell_0!$) limiting the possible group structures to a finite number.

Order of a Group Element

A group element $(\mathcal{G})_\ell$ has order n_ℓ where $n_\ell>0$ is the smallest integer such that

$$(\mathcal{G})_\ell^{n_\ell}=(1) \tag{A.6}$$

with

$$\frac{\ell_0}{n_\ell} = \text{positive integer} \tag{A.7}$$

The set

$$(\mathcal{G})_\ell = \{(\mathcal{G})_\ell^{n} \mid n=1,2,\ldots,n_\ell\} \tag{A.8}$$

is itself a group of order n_ℓ called the period of $(\mathcal{G})_\ell$,and is the subgroup of $\mathcal{G}$ of smallest order containing the element $(\mathcal{G})_\ell$.

Class (Similar Class)

Elements $(\mathcal{G})_\ell$ and $(\mathcal{G})_{\ell'}$ of $\mathcal{G}$ are called similar if an element $(\mathcal{G})'$ in $\mathcal{G}$ exists $\ni$

$$(\mathcal{G})'\,(\mathcal{G})_{\ell'}\,(\mathcal{G})'^{-1} = (\mathcal{G})_\ell \tag{A.9}$$

Furthermore if $(\mathcal{G})_{\ell''}$ is similar to $(\mathcal{G})_\ell$ (say via element $(\mathcal{G})''$) then $(\mathcal{G})_{\ell''}$ is similar to $(\mathcal{G})_{\ell'}$. This similarity relation can be used to separate a group into classes , i.e. sets of elements which are similar to one another. Elements in such classes are sometimes referred as equivalent or conjugate (although this invites confusion with complex conjugate) . The term "similar" is taken directly from the definition of similar matrices. All elements in the same class have the same order. For commutative groups each element forms a single-element class . All groups have (1) as a class by itself.

Invariant Subgroup

For $\mathcal{G}_1$ (order ℓ_1) a subgroup of $\mathcal{G}$ (order ℓ_0) form the set of elements (or complex)

$$(\mathcal{G})_\ell\, \mathcal{G}_1\, (\mathcal{G})_\ell^{-1} \equiv \left\{ (\mathcal{G})_\ell (\mathcal{G}_1)_{\ell'} (\mathcal{G})_\ell^{-1} \mid \ell'=1,2,\ldots,\ell_1 \right\} \tag{A.10}$$

for any $\ell = 1.2,\ldots\ell_0$. This is itself a subgroup of $\mathcal{G}$. If for all $(\mathcal{G})_\ell$ we have

$$(\mathcal{G})_\ell\, \mathcal{G}_1\, (\mathcal{G})_\ell^{-1} = \mathcal{G}_1 \tag{A.11}$$

(i.e. all such subgroups are the same , the group elements of $\mathcal{G}_1$ merely being permuted with one another) , then $\mathcal{G}_1$ is called an invariant subgroup or self-similar subgroup or normal divisor of $\mathcal{G}$ .Since we also have

$$(\mathcal{G})_\ell\, \mathcal{G}_1 = \mathcal{G}_1\, (\mathcal{G})_\ell \quad \text{(left coset= right coset)} \tag{A.12}$$

one can regard an invariant subgroup as one which as a set commutes with every element of $\mathcal{G}$. If none of its invariant subgroups are commutative it is called semisimple.

Factor group

For an invariant subgroup $\mathcal{G}_1$ consider the cosets as in (A.12) as elements of a group known as the factor group or quotient group $\mathcal{G}/\mathcal{G}_1$ with order ℓ_0/ℓ_1 , the index of $\mathcal{G}_1$.

Homomorphism

A group $\mathcal{G}$ is homomorphic to another group $\mathcal{G}'$ if each element of $\mathcal{G}$ can be mapped into a (corresponds to) an element of $\mathcal{G}'$ with group operation preserved . This is distinguished from isomorphism in that several elements of $\mathcal{G}$ may correspond to single element of $\mathcal{G}'$. So homomorphism is a one way mapping whereas isomorphism is symmetrical (i.e. homeomorphic in both directions or one-to-one). Actually equal numbers of elements of $\mathcal{G}$ are mapped onto each element of $\mathcal{G}'$. The set that maps onto the identity of $\mathcal{G}'$ is itself a group, say $\mathcal{G}_s$,which is an invariant subgroup of $\mathcal{G}$. $\mathcal{G}'$ is also isomorphic to the factor group $\mathcal{G}/\mathcal{G}_s$.

Direct Product of Groups

Let $\mathcal{G}$ contain proper subgroups $\mathcal{G}_n$, for $1 \le n \le n_0$, each of order ℓ_n . We can write $\mathcal{G}_n$ as the direct product of the subgroups

$$\mathcal{G} = \mathcal{G}_1 \otimes \mathcal{G}_2 \otimes \cdots \otimes \mathcal{G}_{n_0} \equiv \bigotimes_{n=1}^{n_o} \mathcal{G}_n \tag{A.13}$$

provided the elements of different subgroups commute and every element of $\mathcal{G}$ can be written in exactly one way as

$$(\mathcal{G})_\ell = (\mathcal{G}_1)_{\ell_1'} (\mathcal{G}_2)_{\ell_2'} \cdots (\mathcal{G}_{n_o})_{\ell_{n_o}'} \tag{A.14}$$

(with order unimportant since these commute). The $\mathcal{G}_n$ are called direct factors and are invariant subgroups , the only common element being the identity . Note that we can also write

$$\mathcal{G}_n \otimes \mathcal{G}_{n'} = \mathcal{G}_{n'} \otimes \mathcal{G}_n \tag{A.15}$$

This a special form of direct product , not to be confused with the direct product of two arbitrary groups which forms ordered pairs of elements from the two groups as the new elements. As for order we have

$$\ell_0 = \prod_{n=1}^{n_o} \ell_n \tag{A.16}$$

Faithful Representation

Continuing the concept of matrix representation in section II ,note first that this is itself a group . While one can homomorphically map the elements of a group $\mathcal{G}$ of order ℓ_0 to a group of square (dXd) matrices , one can define a faithful matrix representation as a group of ℓ_0 matrices which is isomorphic to $\mathcal{G}$ under matrix (dot product) multiplication . Note that this is not unique since a similarity transformation will map one faithful matrix representation into another group of ℓ_0 matrices . The identity takes the form

$$(1) \rightarrow \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & \cdot & & \\ & & & 1 & \\ 0 & & & & 1 \end{pmatrix} \tag{A.17}$$

and all the matrices are nonsingular.

Dimension of Representation

A group of square dXd matrices has dimension d. This may come for example from rotations in three dimensional space , in which case d=3 is a natural choice.

Unitary Representation

For finite groups every matrix representation is similar to a unitary matrix representation. Let our matrix representation be unitary dXd so that

$$(\mathcal{G})_\ell \rightarrow (\mathcal{G}^{(r)}_{n,m})_\ell \text{ = group element}$$

$$(\mathcal{G}^{(r)}_{n,m})^{-1}_\ell = (\mathcal{G}^{(r)}_{n,m})^{\dagger}_\ell = \text{group element} \qquad \text{(A.18)}$$

$\dagger \equiv$ adjoint = T* $\equiv$ transpose conjugate

Such unitary matrices have both rows and columns as unit vectors with rows and columns conjugate of each other forming a biorthonormal set (dot products zero except for same row and column). As unitary matrices , when dot multiplying a vector a new vector of the same magnitude (length)is produced . Defining eigenvalues and eigenvectors as·

$$(\mathcal{G}^{(r)}_{n,m})_\ell \cdot (x_n)_{\ell,\beta} = \lambda_{\ell,\beta}\,(x_n)_{\ell,\beta}$$

$$(x_n)^*_{\ell,\beta} \cdot (\mathcal{G}^{(r)}_{n,m})_\ell = \lambda_{\ell,\beta}\,(x_n)^*_{\ell,\beta}$$

$$(x_n)_{\ell,\beta} = \text{columns of } (\mathcal{G}^{(r)}_{n,m})_\ell \qquad \text{(A.19)}$$

$$(x_n)^*_{\ell,\beta} = \text{rows of } (\mathcal{G}^{(r)}_{n,m})_\ell$$

$$(x_n)^*_{\ell,\beta_1} \cdot (x_n)_{\ell,\beta_2} = 1_{\beta_1,\beta_2} \quad \text{(biorthonotmal)}$$

we then have the general properties

$$|\lambda_{\beta,\ell}| = 1 \ , \ \lambda^{-1}_{\beta,\ell} = \lambda^*_{\beta,\ell} \text{ for all } \beta , \ell$$

$$|\det((\mathcal{G}^{(r)}_{n,m})_\ell)| = \prod_{\beta=1}^{d} |\lambda_{\beta,\ell}| = 1 \text{ for all } \ell \qquad \text{(A.20)}$$

$$|\mathrm{tr}((\mathcal{G}^{(r)}_{n,m})_\ell)| \le \sum_{\beta=1}^{d} |\lambda_{\beta,\ell}| = d \text{ for all } \ell$$

Note that for the identity the eigenvalues are all 1 , so the determinant is 1 and the trace is d. If the order of the ℓth element is n_ℓ (a factor of ℓ_0) then we also have

$$(\mathcal{G}^{(r)}_{n,m})^{n_\ell}_\ell = (1_{n,m}) \quad \text{for all } \ell$$

$$\lambda^{n_\ell}_{\beta,\ell} = 1 \text{ for all } \beta, \ell \qquad \text{(A.21)}$$

restricting the eigenvalues to certain roots of 1.

If the unitary representation is also real (real matrices) then the eigenvectors and eigenvalues occur in conjugate pairs (except for the case of real eigenvectors and associated real eigenvalues). In this case the determinant and trace are also real.

Eigenvalues of Representation

Noting that similar matrices have the same eigenvalues , then for finite groups every matrix representation has the same constraints on eigenvalues as in (A.20) . Since classes of $\mathcal{G}$ have all elements similar , then the corresponding matrix representations (in the same dimension d) all have the same eigenvalues . The same applies to functions of (only) the eigenvalues such as trace (or character) and determinant.

Character of a representation

One of the convenient eigenvalue-related properties of a matrix representation is the character or trace as

$$\chi((\mathcal{G}^{(r)}_{n,m})_\ell) = \sum_{n=1}^{d} \mathcal{G}^{(r)}_{n,n;\ell} = \sum_{\beta=1}^{d} \lambda_{\beta,\ell} \tag{A.22}$$

Similar sets of representations have the same set of characters. Similar elements of the representation have the same eigenvalues and hence the same character . So dividing the representation into classes (sets of similar elements) we have one character for each class and hence at most v separate characters χ_ℓ where v is the number of classes. More generally one can speak of v separate sets of d eigenvalues . Note that different representations (not similar ones) can have different characters (distinguished by superscript) . For example , one can have representations with different dimensions d. Note also that

$$\chi((\mathcal{G}^{(r)}_{n,m})^{-1}_\ell) = \chi^*((\mathcal{G}^{(r)}_{n,m})_\ell) \tag{A.23}$$

Reduction of Representation

The representation $\left\{ (\mathcal{G}^{(r)}_{n,m})_\ell \mid \ell'=1,2,\ldots,\ell_0 \right\}$ of a group $\mathcal{G}$ is said to be reducible or decomposable if it can be put in the form (by a similarity transformation)

$$\begin{aligned}
(\mathcal{G}_{n,m})_\ell &= \begin{pmatrix} (\mathcal{G}^{(1)}_{n,m})_\ell & & & 0 \\ & (\mathcal{G}^{(2)}_{n,m})_\ell & & \\ & & \ddots & \\ 0 & & & (\mathcal{G}^{(u_0)}_{n,m})_\ell \end{pmatrix} \\
&= (\mathcal{G}^{(1)}_{n,m})_\ell \oplus (\mathcal{G}^{(2)}_{n,m})_\ell \oplus \cdots \oplus (\mathcal{G}^{(u_0)}_{n,m})_\ell \\
&= \bigoplus_{u=1}^{u_0} \mathcal{G}^{(u)}_{n,m})_\ell
\end{aligned} \tag{A.24}$$

$\oplus \equiv$ direct sum

where each diagonal block $(\mathcal{G}^{(u)}_{n,m})_\ell$ is square $d_u \times d_u$ with

$$d=\sum_{u=1}^{u_0} d_u \tag{A.25}$$

Note that the same block decomposition (same ordered set of d_u) is required to apply for all ℓ. Then each matrix group

$$\mathcal{G}^{(u)}= \left\{ (\mathcal{G}^{(u)}_{n,m})_\ell \mid \ell=1,2,\ldots,\ell_0 \right\} \tag{A.26}$$

is itself a representation of $\mathcal{G}$ of dimension d_u. If the representation cannot be decomposed into two or more such diagonal blocks by a similarity transformation the representation is called irreducible. The representation is fully reducible or fully decomposable to the form in (A.24) when each representation as in (A.26) is irreducible. Among the irreducible representations some may not be distinct in that they may be similar (applying of course potentially to representations with the same dimension). Then by a similarity transformation such representations can be taken as the same and we can write

$$(\mathcal{G}_{n,m})_\ell = \left[\bigoplus_{v=1}^{v_1} (\mathcal{G}^{(1)}_{n,m})_\ell \right] \oplus \left[\bigoplus_{v=2}^{v_2} (\mathcal{G}^{(2)}_{n,m})_\ell \right] \oplus \cdots \oplus \left[\bigoplus_{v=1}^{v_{u_1}} (\mathcal{G}^{(u_1)}_{n,m})_\ell \right]$$

$$= \bigoplus_{u=1}^{u_1} \left[\bigoplus_{v=1}^{v_u} (\mathcal{G}^{(u)}_{n,m})_\ell \right] \tag{A.27}$$

$v_u \equiv$ multiplicity of each $(\mathcal{G}^{(u)}_{n,m})_\ell$

$u_1 \equiv$ number of distinct irreducible representations

$$d=\sum_{u=1}^{u_1} v_u\, d_u$$

$$\mathcal{X}((\mathcal{G}_{n,m})_\ell) = \sum_{u=1}^{u_1} v_u\, \mathcal{X}((\mathcal{G}^{(u)}_{n,m})_\ell)$$

$$\det((\mathcal{G}_{n,m})_\ell) = \prod_{u=1}^{u_1} v_u \left[\det((\mathcal{G}^{(u)}_{n,m})_\ell)\right]^{v_u}$$

The character $\mathcal{X}((\mathcal{G}_{n,m})_\ell)$ of a reducible representation is called a compound character.

The characters $\mathcal{X}((\mathcal{G}^{(u)}_{n,m})_\ell)$ of the irreducible representations are called primitive characters

or simple characters. Note that the similarity transformation of the matrix representation can be thought of as an appropriate selection of the d coordinates (variables) which the matrices multiply to give the various symmetries (invariances) associated with the group representations.

Orthogonality of Representations

For all non-similar irreducible unitary representations

$$\sum_{\ell=1}^{\ell_0} \mathcal{G}^{(u)}_{n,m;\ell}\, \mathcal{G}^{(u')*}_{n',m';\ell} = \frac{\ell_0}{d_u} 1_{u,u'}\, 1_{n,n'}\, 1_{m,m'} \tag{A.28}$$

Character Orthogonality

The primitive characters of the non-singular irreducible representations satisfy

$$\frac{1}{\ell_0}\sum_{\ell=1}^{\ell_0} \mathcal{X}((\mathcal{G}^{(u)}_{n,m})_\ell)\ \mathcal{X}^*((\mathcal{G}^{(u')}_{n,m})_\ell) = 1_{u,u'} \tag{A.29}$$

This is applied to a reducible representation to find (with symbols as in (A.27))

$$\frac{1}{\ell_0}\sum_{\ell=1}^{\ell_0} \mathcal{X}((\mathcal{G}_{n,m})_\ell)\ \mathcal{X}^*((\mathcal{G}_{n,m})_\ell) = \frac{1}{\ell_0}\sum_{\ell=1}^{\ell_0} |\mathcal{X}((\mathcal{G}_{n,m})_\ell)|^2$$

$$= \sum_{u=1}^{u_1} v_u^2 \geq 1 \tag{A.30}$$

with equality iff the representation is irreducible . To find the number of times v_u an irreducible representation is found in a matrix representation we can use

$$v_u = \frac{1}{\ell_0}\sum_{\ell=1}^{\ell_0} \mathcal{X}((\mathcal{G}^{(u)}_{n,m})_\ell)\ \mathcal{X}^*((\mathcal{G}_{n,m})_\ell) = \frac{1}{\ell_0}\sum_{\ell=1}^{\ell_0} \mathcal{X}^*((\mathcal{G}^{(u)}_{n,m})_\ell)\ \mathcal{X}((\mathcal{G}_{n,m})_\ell) \tag{A.31}$$

The above formulas can be also written in forms which explicitly account for the ν classes of $\mathcal{G}$ Noting that each class has the same character , say $\mathcal{X}_i$ (for $i=1,2,\ldots\nu$) one can sum over the classes with weights ν_i as the class multiplicity (number of elements). These properties can be summarized in a character table in which the columns are the characters for each element of $(\mathcal{G}_{n,m})_\ell$ and its irreducible representations $(\mathcal{G}^{(u)}_{n,m})_\ell$, each representation corresponding to a row. Note that not all irreducible representations need be faithful.

Regular Matrix Representations of an Abstract Group

An abstract group

$$\mathcal{G} = \left\{ (\mathcal{G})_\ell \ \mid \ell = 1,2,\ldots \ell_0 \right\} \tag{A.32}$$

admits a regular representation in terms of $\ell_0 X \ell_0$ permutation matrices (dimension ℓ_0) This is given by

$$\mathcal{G}_{n,m;\ell} = \begin{cases} 1 \text{ if } (\mathcal{G})_n^{-1}(\mathcal{G})_\ell(\mathcal{G})_m = (1) \\ 0 \text{ otherwise} \end{cases}$$

$$\mathcal{X}((\mathcal{G}_{n,m})_\ell) = \begin{cases} \ell_0 \ \text{ for } ((\mathcal{G})_{n,m})_\ell = (1_{n,m}) \\ 0 \text{ otherwise} \end{cases} \tag{A.33}$$

Irreducible Representations of an Abstract Group

An abstract group

$$\mathcal{G} = \left\{ (\mathcal{G})_\ell \ \mid \ell = 1,2,\ldots \ell_0 \right\} \tag{A.34}$$

can admit a number of irreducible matrix representations . Let u_{max} be the number of distinct (non-similar) irreducible representations .Then we have

u_{max}=number of classes (of similar elements) in $\mathcal{G}$

$$\ell_o = \sum_{u=1}^{u_{max}} d_u^2 \qquad (A.35)$$

d_u = dimension of uth irreducible representation (distinct)

$\frac{\ell_o}{d_u}$ = integer (positive) for $u=1,2...u_{max}$

Also the regular representation contains the uth irreducible representation exactly d_u times.

Appendix B. Properties of Infinite Groups

Letting the number of group elements ℓ_o become infinite introduces some new features in groups. In particular the order n_ℓ of a group element (as in (A.60 need not be finite or even exist , i.e. as we take an element $(\mathcal{G})_\ell$ and keep multiplying it by itself we get group elements of the form $(\mathcal{G})_\ell^m$ but no value of m may give the identity (1) . This can be referred to as a lack of closure. In a matrix representation this admits cases in which the group does not have a unitary representation , the magnitudes of the eigenvalues perhaps diverging or tending to zero. Furthermore the representation may not be fully reducible or fully decomposable to the direct sum form in (A.24) and (A, 27) . It may still be reducible in the sense that there remain matrix blocks on one side (upper or lower, but not both) of the diagonal blocks.

Discrete Infinite Groups

A group may have an infinite number of elements as

$$\mathcal{G} = \{(\mathcal{G})_\ell \mid \ell = 1,2\ldots,\infty\} \tag{B.1}$$

This kind of group can also be referred to as a countable or denumerable .Examples include the discreet space groups (section VIII) and discrete similarity groups (section IX).

Continuous (Lie) Groups

Introducing the concept of an infinitesimal change in a group element to give another group element we have a continuous group or a Lie group. A g-parameter continuous group has elements as functions of g real continuously varying parameters as

$$\mathcal{G} = \{\mathcal{G}(a_1,a_2\ldots,a_g)\} \tag{B.2}$$

The number g can be finite or infinite. The g parameters are called essential if the group elements cannot be labelled with a smaller number of real parameters. The range of the parameters may be finite or infinite . If the range is finite for all the parameters the group manifold is closed or compact.

Compact Continuous Groups

A compact continuous group (g real parameters of bounded variation) has certain properties of finite groups . In particular it is similar to a unitary group so that a matrix representation has eigenvalues and determinant with unit magnitude. Every representation is fully reducible or fully decomposable to a direct sum of irreducible representations . The orthogonality relations as in (A.28) through (A.31) also go through as before except that the sum is replaced by an integral over the group manifold with a multidimensional volume element corresponding to the parameters being integrated . Examples of such groups are C_∞ (or O_2^+) and O_3^+ consisting of proper rotations in 2 and 3 dimensions , respectively , as discussed in section VI.

Appendix C. Illustrative Example : Antenna Array with C_3, C_{3a} ,and C_{3r} Symmetry.

In order to put some of the forgoing group theory into perspective let us consider a simple example , an antenna array (introduced in section VI) . Schematically indicated in fig. C.1 . Here we have three antennas located at common radius Ψ_0 but at three angles

$$\phi_n = \frac{2\ell\pi}{3},\ \ell = 1,2,3 \tag{C.1}$$

These antennas need not be symmetric with respect to the planes P_ℓ defined by these angles, but can be oriented at some directions $\mathbf{D}_\ell$ at common angles α with respect to these planes , perhaps with common z components as well. Each orientation can characterize the direction of some antenna element or the pattern (not necessarily symmetric about the $\mathbf{D}_\ell$). In any event the antenna array has C_3 symmetry , invariance on successive rotation by $2\pi/3$.

The group can be expressed as a 1-dimensional matrix representation (irreducible and commutative with three classes)

$$C_3 \rightarrow \left\{ e^{j\frac{2\pi\ell}{3}}\ \middle|\ \ell = 1,2,3 \right\} \tag{C.2}$$

with characters the same as the group elements . There is also the commonly used 2-dimensional form (as in section VI) given by

$$C_3 \rightarrow \left\{ \begin{pmatrix} \cos(\frac{2\ell\pi}{3}) & -\sin(\frac{2\ell\pi}{3}) \\ \sin(\frac{2\ell\pi}{3}) & \cos(\frac{2\ell\pi}{3}) \end{pmatrix} \middle|\ \ell = 1,2,3 \right\}$$

$$(C_3)_1 = (C_3)_2^{-1} \rightarrow \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

$$(C_3)_2 = (C_3)_1^{-1} \rightarrow \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \tag{C.3}$$

$$(C_3)_3 = (1) \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (1_{n,m})$$

This gives the following character table

Table C.1 : Character Table for Faithful Two -Dimensional Representation of C_3 (Three Classes)

	$(1)=(C_3)_3$	$(C_3)_1$	$(C_3)_2$
C_3 (d=2)	2	-1	-1
$C_3^{(1)}(d_1=1)$	1	$e^{j\frac{2\pi\ell}{3}}$	$e^{j\frac{4\pi\ell}{3}}$
$C_3^{(2)}(d_2=1)$	1	$e^{-j\frac{2\pi\ell}{3}}$	$e^{-j\frac{4\pi\ell}{3}}$

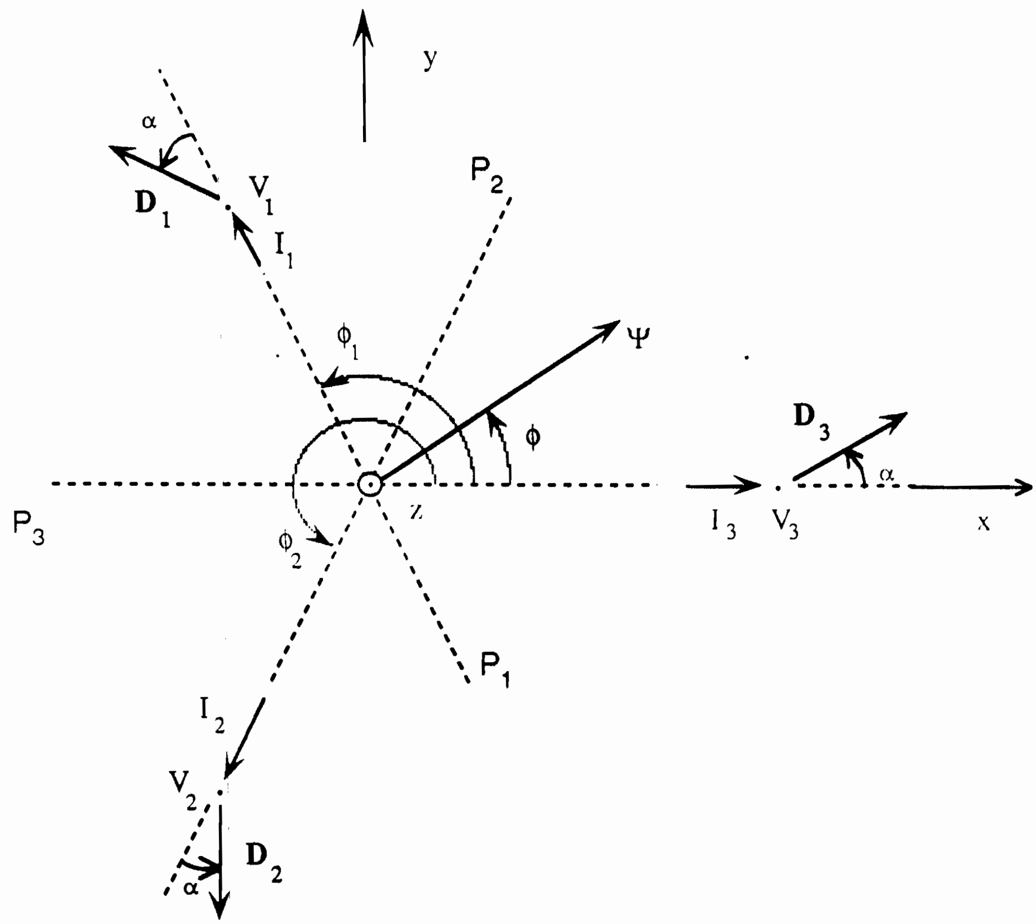


Figure. C.1 Example Antenna Array

Note from (A.29) through (A.31) that the two dimensional representation is fully decomposed with

$u_1 = 2\ ,\ v_1 = v_2 = 1$

$d\ = 2\ ,\ d_1 = d_2 = 1$

$$(C_3)_\ell = (C_3^{(1)})_\ell \oplus (C_3^{(2)})_\ell \ \rightarrow \begin{pmatrix} e^{j\frac{2\pi\ell}{3}} & 0 \\ 0 & e^{-j\frac{2\pi\ell}{3}} \end{pmatrix}$$

$$(C_3^{(1)})_\ell \rightarrow e^{j\frac{2\pi\ell}{3}} \qquad \text{(C.4)}$$

$$(C_3^{(2)})_\ell \rightarrow e^{-j\frac{2\pi\ell}{3}}$$

One can also arrive at this by constructing the eigenvectors of the matrices in (C.3) and form

$$(X_{n,m}) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & j \\ 1 & -j \end{pmatrix} \qquad \text{(C.5)}$$

the rows of which are the eigenvectors common to all three matrices. Noting that

$$(X_{n,m})^{-1} = (X_{n,m})^{\dagger} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -j & j \end{pmatrix} \qquad \text{(C.6)}$$

we have the representation

$$(X_{n,m})\cdot\begin{pmatrix} \cos(\frac{2\ell\pi}{3}) & -\sin(\frac{2\ell\pi}{3}) \\ \sin(\frac{2\ell\pi}{3}) & \cos(\frac{2\ell\pi}{3}) \end{pmatrix}\cdot(X_{n,m})^{-1} = \begin{pmatrix} e^{j\frac{2\pi\ell}{3}} & 0 \\ 0 & e^{-j\frac{2\pi\ell}{3}} \end{pmatrix} \text{ for } \ell=1,2,3 \qquad \text{(C.7)}$$

from an explicit similarity transformation. Note that the eigenvectors used in (C.5) are complex. Thus the simplicity obtained in the representation in (C.7) is obtained at the expense of rotation to complex coordinates.

Next increase the symmetry by setting $\alpha=0$ and requiring each antenna in the array to be symmetric with respect to the associated symmetry plane P_ℓ . This, of course makes each of the antenna patterns symmetric with respect to the associated symmetry plane. These symmetry planes are all axial (contain the 3-fold rotation axis, the z axis) giving C_{3a} symmetry.

There are various ways to organize the six elements of this group . In a two-dimensional representation let

$$(C_{3a})_\ell = \begin{cases} (C_3)_\ell & \text{for } \ell=1,2,3 \\ (R)_{\ell-3} & \text{for } \ell=4,5,6 \end{cases} \qquad \text{(C.8)}$$

where $(R)_{\ell'}$ represents reflection through $P_{\ell'}$ as

$$(R)_1 = (C_3)_1\,(R_3)\,(C_3)_1^{-1} \ \rightarrow \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

$$(R)_2 = (C_3)_2\,(R_3)\,(C_3)_2^{-1} \ \rightarrow \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \qquad \text{(C.9)}$$

$$(R)_2 \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and that these three elements form a class . Furthermore we have

$$(R)_{\ell'} \cdot (C_3)_1 \cdot (R_{\ell'})^{-1} = (C_3)_2 \tag{C.10}$$

so that $(C_3)_1$ and $(C_3)_2$ now form a class in (C_{3a}) . The group (C_{3a}) has then three classes.

The (C_{3a}) group has 4 proper subgroups

$$\begin{aligned} &C_3 = \{(C_3)_1 , (C_3)_2 ,(1)\} \\ &(C_3)_1 (C_3)_2 = (C_3)_2 (C_3)_1 = (1) \\ &(R)_{\ell'} =\{ (1), (R)_{\ell'} \} \\ &(R)^2_{\ell'} =(1) \qquad \text{for } \ell' =1,2,3 \end{aligned} \tag{C.11}$$

Of course one of these reflection groups is needed with C_3 to generate the full C_{3a} group due to the similarity of the $(R)_{\ell'}$. The character table follows.

Table C.2 Character Table for the Faithful Two-Dimensional Representation of C_{3a} (Three Classes)

	$(1) =(C_3)_3$	$(C_3)_1$, $(C_3)_2$ (weight 2)	$(R)_{\ell'}$ (weight 3)
$C_{3a}(d=2)$	2	-1	0

How interesting! Per (A.30) this is , as it stands , an irreducible representation of the C_{3a} group. An alternate form is given by application of a similarity transformation via $(X_{n,m})$ from (C.5) as used in (C.7) giving

$$(C_{3a})_{\ell} \rightarrow \begin{cases} \begin{pmatrix} e^{j\frac{2\pi\ell}{3}} & 0 \\ 0 & e^{-j\frac{2\pi\ell}{3}} \end{pmatrix} & \text{for } \ell=1,2,3 \\ \begin{pmatrix} 0 & e^{-j\frac{2\pi\ell'}{3}} \\ e^{j\frac{2\pi\ell'}{3}} & 0 \end{pmatrix} & \text{for } \ell'=\ell-3=1,2,3 \end{cases} \tag{C.12}$$

as a simpler looking form of a matrix representation.

The (C_{3a}) group is also isomorphic to the diehedral group D_3 . This can be seen by replacing reflection through $P_{\ell'}$ by rotation of the figure out of the z=0 plane about an axis formed by the intersection of P_{ℓ} with the z=0 plane. One can go even further by considering reflection through the z=0 plane (z→-z) with a group element $(R)_t$ to give a group C_{3at} (or D_{3t}) of order 12. This kind of reflection commutes with all the previous elements . Physically this additional symmetry corresponds to making each antenna in the antenna also symmetric with respect to the z=0 plane (and thereby the pattern contains such symmetry depending on the excitation of the array elements)

Let us now shift gears and consider another kind of symmetry in our antenna array problem. As indicated in fig. C.1 each of the three antennas is driven by a voltage V_{ℓ} and a current I_{ℓ} . These are related via

$$\begin{aligned} &(\tilde{V}_n(s)) = (\tilde{Z}_{n,m}(s))\cdot(\tilde{I}_n(s)) \\ &(\tilde{Z}_{n,m}(s)) \equiv \text{impedance matrix (3x3)} \\ &(\tilde{Y}_{n,m}(s)) \equiv (\tilde{Z}_{n,m}(s))^{-1} \equiv \text{admittance matrix} \end{aligned} \tag{C.13}$$

As is well known in circuit theory reciprocity implies

$$(\tilde{Z}_{n,m}(s))^T = (\tilde{Z}_{n,m}(s))\ ,\ \ (\tilde{Y}_{n,m}(s))^T = (\tilde{Y}_{n,m}(s)) \tag{C.14}$$

This kind of symmetry is not in the geometry but comes from the Maxwell equations (section IV).

C_3 symmetry implies that as one successively rotates the array by $2\pi/3$, the labels on the voltage and current can be cyclically permuted with no change in the impedance matrix. This means that the matrix elements $Z_{n,m}$ are only a function of m-n (modulo 3). This is a special kind of circulant matrix. As discussed in [4.15] in a circulant matrix each row is the same as the previous row shifted one to the right (with the last element going to the beginning). Furthermore the matrix symmetry makes the matrix bicirculant such that the row shifting properly also applies to the columns. The matrix elements are then only a function of $|m-n|$ (modulo 3). For the C_3 example there are only two independent elements of the impedance matrix. All diagonal elements are the same as $\tilde{Z}_{1,1}$, and all off-diagonal elements are the same as $\tilde{Z}_{1,2}$. Note that no reflection symmetry in the array is assumed for this result (i.e. in general $\alpha \neq 0$ and the $\mathbf{D}_\ell$ can have z components in fig. C.1). However, the electrical symmetry is such that it does not matter whether one counts around the antenna ports in a $+\phi$ or $-\phi$ sequence. This reversal of port ordering is like a $(P)_\ell$ reflection symmetry, but not a reflection in the geometry or antenna pattern. This inclusion of reciprocity can then be designated as C_{3r} symmetry.

Applying the results of [4.15] we have

$$\begin{aligned}
&(\tilde{Z}_{C_{n,m}}(s))\cdot(x_n)_\beta = (x_n)_\beta \cdot(\tilde{Z}_{C_{n,m}}(s)) = \tilde{Z}_\beta(s)(x_n)_\beta \\
&(x_n)_{\beta_1}\cdot(x_n)_{\beta_2} = 1_{\beta_1,\beta_2} \\
&\tilde{Z}_\beta(s) = \tilde{Z}_{1,1}(s) + 2\ \tilde{Z}_{1,2}(s)\cos(\tfrac{2\pi}{3}) \\
&\tilde{Z}_1(s) = \tilde{Z}_2(s) = \tilde{Z}_{1,1}(s) - \tilde{Z}_{1,2}(s) \\
&\tilde{Z}_3(s) = \tilde{Z}_{1,1}(s) + 2\ \tilde{Z}_{1,2}(s)
\end{aligned} \tag{C.15}$$

exhibiting the eigenvalue degenaracy (only two distinct ones). Note for C_3 symmetry the 3x3 impedance matrix has only two distinct elements. This degeneracy also affects the eigenvectors which can be taken in a real form as

$$\begin{aligned}
(x_n)_1 &= \sqrt{\tfrac{2}{3}}\begin{pmatrix}\cos(\frac{2\pi}{3})\\ \cos(\frac{4\pi}{3})\\ 1\end{pmatrix} = \frac{1}{\sqrt{6}}\begin{pmatrix}-1\\-1\\2\end{pmatrix} \\
(x_n)_2 &= \sqrt{\tfrac{2}{3}}\begin{pmatrix}\cos(\frac{2\pi}{3})\\ \cos(\frac{4\pi}{3})\\ 0\end{pmatrix} = \frac{1}{\sqrt{6}}\begin{pmatrix}-1\\-1\\0\end{pmatrix} \\
(x_n)_3 &= \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}
\end{aligned} \tag{C.16}$$

The first two eigenvectors are not unique due to their having the same eigenvalue. The third eigenvector corresponds to a common mode. Refering back to fig. C.1 these eigenvectors can be thought of as patterns for the voltages and currents exciting the array with impedances related to the eigenvalues (eigenimpedances).

So how do we relate this 3X3 matrix and associated 3-dimensional eigenvectors to the previous group structures? One way to do this is to think of what these electrical symmetries represent . The invariances are to certain interchanges of the labels (1,2,3) on the antenna ports . These can be written as permutation matrices . For our matrix representation then let us take 3x3 permutation matrices. For rotation we have

$$\begin{aligned} (C_3)_1 &\rightarrow \begin{pmatrix} 0&0&1 \\ 1&0&0 \\ 0&1&0 \end{pmatrix} \\ (C_3)_2 &\rightarrow \begin{pmatrix} 0&1&0 \\ 0&0&1 \\ 1&0&0 \end{pmatrix} \\ (C_3)_3=(1) &\rightarrow \begin{pmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{pmatrix} \end{aligned} \tag{C.17}$$

These correspond to rotation around the array in the sense that $(C_3)_1$ moves vector components (of volltage ,current, eigenvectors) around the array as $1\rightarrow 2$, $2\rightarrow 3$, $3\rightarrow 1$ and similarly for other such matrices . As these stand in (C.17) they give an acceptable matrix representation of C_3 . (This can readily be generalized to C_N for an N-element array.) Similarly the reciprocity condition gives a set of reciprocity -reflection matrices as

$$\begin{aligned} (R)_1 &\rightarrow \begin{pmatrix} 1&0&0 \\ 0&0&1 \\ 0&1&0 \end{pmatrix} \\ (R)_2 &\rightarrow \begin{pmatrix} 0&0&1 \\ 0&1&0 \\ 1&0&0 \end{pmatrix} \\ (R)_3 &\rightarrow \begin{pmatrix} 0&1&0 \\ 1&0&0 \\ 0&0&1 \end{pmatrix} \end{aligned} \tag{C.18}$$

Here the reflection is of antenna-port labels (not geometry) about the plane $P_{\ell'}$ in fig, C.1. (Again this can be generalized to an N-element array.) Now we have the 6 elements in a 3-dimensional representation with 2N elements.) Again the $(P)_{\ell'}$ form as class as do $(C_3)_1$ and $(C_3)_2$. The character table follows.

Table C.3 Character Table for Faithful Three-Dimensional Representation of C_{3r} (Three Classes)

	$(1) = (C_3)_3$	$(C_3)_1$, $(C_3)_2$ (weight 2)	$(R)_{\ell'}$ (weight 3)
$C_{3r}(d=3)$	3	0	1
$C_{3r}^{(1)}(d_1=1)$	1	1	1
$C_{3r}^{(2)}(d_2=1)$	2	-1	0

Note that this consistent with (A.29) through (A.31) with

$$\begin{aligned} &u_1 = 2 \text{ , } v_1 = v_2 = 1 \\ &d = 3 \text{ , } d_1 = 1 \text{ , } d_2 = 1 \\ &(C_{3r})_\ell \rightarrow (C_{3r}^{(1)})_\ell \oplus (C_{3r}^{(2)})_\ell \end{aligned} \tag{C.19}$$

$(C_{3r}^{(1)})_\ell = (1) \rightarrow 1$ for ℓ =1 through 6

$(C_{3r}^{(2)})_\ell = (C_{3a})_\ell \rightarrow$ 6 matrices (2X2) in (C.3) and (C.9) or in (C.12)

The irreducible representation $(C_{3r}^{(1)})_\ell$ is a homomorphism of 6 elements onto 1 (the group consisting of the identity). As such this is not an isomorphism or faithful representation (a faithless representation ?). However the irreducible representation $(C_{3r}^{(2)})_\ell$ is a faithful 6-element representation with the same 2-dimensional representation as (C_{3a}) discussed previously.

The similarity transformation to this form is found by constructing the common (right) eigenvectors of the (C.17) matrices in cyclic form (with first element as 1 for later convenience) as

$$(y_n)_\beta = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ e^{-j\frac{2\pi\beta}{3}} \\ e^{-j\frac{4\pi\beta}{3}} \end{pmatrix}, \quad \lambda_\beta = e^{-j\frac{2\pi\beta}{3}} \quad , \beta = 1,2,3$$

$$(y_n)_{\beta_1} \cdot (y_n)^*_{\beta_2} = 1_{\beta,\beta'} \tag{C.20}$$

the left eigenvalues being conjugate to the right. Form the matrix from the left eigenvectors as rows (with order for later convenience)

$$(Y_{n,m}) = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{j\frac{2\pi}{3}} & e^{-j\frac{2\pi}{3}} \\ 1 & e^{-j\frac{2\pi}{3}} & e^{j\frac{2\pi}{3}} \end{pmatrix} = \begin{pmatrix} (y_n)^*_3 \\ (y_n)^*_1 \\ (y_n)^*_2 \end{pmatrix}$$

$$(Y_{n,m})^{-1} = (Y_{n,m})^\dagger = ((y_n)_3 , (y_n)_1 ,(y_n)_2) \tag{C.21}$$

Applying this to the matrices in (C.17) gives

$$(C_3)_\ell \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{j\frac{2\pi\ell}{3}} & 0 \\ 0 & 0 & e^{-j\frac{2\pi\ell}{3}} \end{pmatrix} \quad \text{for } \ell = 1,2,3 \tag{C.22}$$

and to those in (C.18) gives

$$(R)_{\ell'} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & e^{-j\frac{2\pi(\ell'-1)}{3}} \\ 0 & e^{j\frac{2\pi(\ell'-1)}{3}} & 0 \end{pmatrix} \quad \text{for } \ell' = \ell\text{-}3 = 1,2,3 \tag{C.23}$$

exhibiting the simultaneous block diagonalization of all the matrices in the group to form a fully reduced representation . Note that $C_{3r}^{(1)}$ occupies the upper left 1x1 block while $C_{3r}^{(2)}$ occupies the lower right 2x2 block . The reflection matrices (ℓ =4,5,6) in $C_{3r}^{(2)}$ are not in the same order as in (C.9) and (C.12) , but this is not an essential difference since the set of group elements is the same.

One can find all the irreducible representations of the C_{3a} or C_{3r} group based on the regular representation (6x6) as in (A.33) . In this case we have from (A.35)

$u_{max} = 3$ (number of classes)

$$6 = \sum_{u=1}^{3} d_u^2 \tag{C.24}$$

$d_1=1$, $d_2=1$, $d_3=2$

The character table follows.

Table C.4 Character Table for Irreducible Representation of Abstract C_{3a} or C_{3r} Group

	$(1) = (C_3)_3$	$(C_3)_1$, $(C_3)_2$ (weight 2)	$(R)_{\ell'}$ (weight 3)
{1} ($d_1=1$)	1	1	1
{1,-1} ($d_2=1$)	1	1	-1
$C_{3a}^{(1)}$ or $C_{3r}^{(2)}$ ($d_3=2$)	2	-1	0

Here there are two non-faithful irreducible representations . The identity group has already been encountered in which all elements are homomorphically mapped onto a single element 1. One can also map the $(R)_{\ell'}$ elements onto -1 to obtain a two-element group.

So $C_{3r}^{(2)}$ (as well as C_{3r}) is isomorphic to C_{3a} , all having the same 2X2 faithful representation. Both are also isomorphic to the symmetric group of degree 3 (all permutations of 3 symbols).

We have thus found an interesting interchange between physics (reciprocity) and geometry. Axial reflection symmetry adjoined to two dimensional rotation symmetry C_3 produces C_{3a} symmetry in the fields radiated by an array. Reciprocity (the usual condition in antenna problems) adjoined to C_3 rotation symmetry produces C_{3r} symmetry in the antenna-port signals. However, both have the same group structure. The reader can note that while the example here is based on C_3 , the procedure applies to C_N , so that C_{Na} and C_{Nr} are isomorphic with 2N elements each. However, they are not isomorphic to the symmetric group of degree N (for N>3) , being isomorphic to a proper subgroup of this.

Bibliography

I.General

[1.1] C.E. Baum,"Emerging Technology for Transient and Broad-Band Analysis and Synthesis of Antennas and Scatterers", Proc. IEEE, pp. 1598-1616, 1976.

[1.2] C.E. Baum,"Where is Electromagnetics Going"? pp. 6-10 in J.A. Kong (ed.), NSF Workshop of Future Directions in Electromagnetics Research, Boston, 1989.

[1.3] D.K. Cohoon,"Reduction of the Cost of Solving an Integral Equation Arising in Electromagnetic Scattering Through The Use of Group Theory", IEEE Tans, AP, pp. 104-107, 1980.

[1.4] M. Carmeli,"Group Analysis of Maxwell's Equations" J. Math, Phys., 1969, pp. 1699-1703.

[1.5] S.R. Cloude, "Lie Groups and Polarimetry" PIERS Proc. 1989, Boston, p.421.

[1.6] G. A. Deschamps,"Polarization Descriptors: Utilization of Symmetries", pp. 493 - 517 in W.M. Boerner (ed.), Inverse Methods in Electromagnetic Imaging, Reidel, 1985.

[1.7] D. Gorenstein,"The Enormous Theorem",Scientific Americam ,Dec.1985, pp104-15.

[1.8] W.Hauser ,"On the Fundamental Equations of Electromagnetism" American Journal of Physics,pp.80-85,1970.

[1.9] Henry and Lonsdale "International Tables for X-Ray Crystallography"(1965)

[1.10] B.Ye Kinber A. Kotlyar,"Use of symmetry in solving diffraction problems". Radio Engineering and Electronic Physics ,pp. 581-587 , 1971.

[1.11] H.N. Kritikos,"Radiation Symmetries of Antenna Arrays" J. Franklin Inst., pp. 283-292, April 1972 .

[1.12] P.L. Overfelt, "Symmetry operators and Nonseparable solutions of the two-dimensional Helmhotz Equation for Electromagnetic Analysis" Electromagnetics, pp. 249-258, 1989.

[1.13] M.Pease ,"Application of Lie Algebraic Theory to Microwave Networks",Advances in Microwaves,ed. L. Young , Academic Press (1966).

[1.14] Paul F. Wacker, "Symmetry Techniques for Direct and Inverse Scattering", pp. 203-222 in W.M. Boerner (ed.), Inverse Methods in Electromagnetic Imaging, Reidel, 1985.

[1.15] P.C. Waterman,"Symmetry,Unitarity,and Geometry in Electromagnetic Scattering",Phys.Rev. D,Vol.3 Nu. 4. , pp 825-39, Feb.1971.

[1.16] V.H. Weston,"Theory of Absorbers in Scattering IEEE Trans. AP", pp. 578-584, 1963.

II. Duality

[2.1] C.E. Baum, "Production of Uniform Electrostatic Fields by a Slotted Conducting Spherical Shell ",Sensor and Simulation Note 304, 1987 and IEEE Trans. EMC, 1988, pp 9-12 .

[2.2] C.E. Baum,"Dual Sources on Boundaries", Sensor and Simulation Note 310, 1988

[2.3] C.E. Baum,"The PARTES Concept in EMP Simulation," Sensor and Simulation Note 260, 1979 and Electromagnetics, 1983, pp. 1-19.

[2.4] H. Bremmer,"Asymptotic Developments and Scattering Theory in Terms of a Vector Combining the Electric and Magnetic Fields" IRETrans.AP, 1956,pp.264-265.

[2.5] B.K. Singaraju andC.E. Baum,"The Combined Field: Boundary Conditions, Integral Representation, Dyadic Green's Functions and Some Theorems and Concepts ",Mathematics Note 38, 1974.

[2.6] L. Silberstein,"Elektromagnetische Grundgleichungen in bivektorieller Behandlung" Annalen der Physik, ser. 4, vol. 22, 1907 pp. 579-586.

[2.7] L. Silberstein,"Nachtrag zur Abhandlung uber Elektromagnetische Grundgleichungen in bivektorieller", Behandlung Annalender Physik, ser. 4, vol. 24, 1907 pp. 783-78.

[2.8] C.E.Baum,"Singularity Expansion of Electromagnetic Fields and Potentials Radiated from Antennas or Scattered from Objectcs in Free Space", Sensor and Simulation Note 179,1973.

III. Reciprocity and Energy Theorems

[3.1] C. Altman and A. Postan,"A Reciprocity Theorem for Magnetoionic Modes", Radio Science, 1971, pp. 483-487.

[3.2] C.E. Baum,"Electromagnetic Reciprocity and Energy Theorem for Free Space Including Sources Generalized to Numerous Theorems, to Combined Fields, and to Complex Frequency Domain", Mathematics Note 33,1973.

[3.3] S. Ballantine, "Reciprocity in Electromagnetic, Mechanical, Acoustical, and Interconnected Systems" , Proc. IRE, 1929, pp. 929-951.

[3.4] J. Brown,"A Generalized Form of the Aerial Reciprocal Theorem", IEE Monograph No. 301R Apr. 1958", pp 472-475.

[3.5] J.R. Carson,"A Generalization of the Reciprocal Theorem" , Bell Sys Tech. Jour., July 1924, pp. 393-399.

[3.6] J.R. Carson, "The Reciprocal Energy Theorem", Bell Sys. Tech. Jour. 1930, pp. 325-331.

[3.7] J.R. Carson, "Reciprocal Theorems in Radio Communication ", Proc. IRE June 1929, pp. 952-95.

[3.8] M.H. Cohen,"Application of the Reaction Concept to Scattering Problems", IRE Trans. AP, 1955, pp 193-199.

[3.9] B. R-S. Cheo,"A Reciprocity Theorem for Electromagnetic Fields with General Time Dependence", IEEE Trans. AP, 1965, pp. 278-284.

[3.10] A.T. DeHoop,"A Reciprocity Relation Between the Transmitting and Receiving Properties of an Antenna ", Appl. Sci. Res. vol. 19, 1968 pp. 90-96.

[3.11] A.T. DeHoop,"A Reciprocity Theorem for the Electromagnetic Field Scattered by an Obstacle", Applied Sci. Res. Section B, Vol. 8, 1960, pp. 135-140

[3.12] G. Goubau,"A Reciprocity Theorem for Nonperiodic Fields" , IRE Trans. AP, 1960, pp.339-342.

[3.13] J.A. Kong and D.K. Cheng,"Modified Reciprocity Theorem for Bianisotropic Media", Proc. IEE, 1970, pp. 349-350.

[3.14] J.A. Kong,"Theorems of Bianisotropic Media", Proc. IEEE 1972, pp. 1036-1046.

[3.15] G.D. Monteath,"Application of the Compensation Theorem to Certain Radiation and Propagation Problems", Proc. IEE, Vol. 98; part 4, #1, Oct. 1951, pp. 23-30.

[3.16] J. Robieux, "General Theorems on the Transmission Coefficient from a Transmitting to a Receiving System", IRE Trans AP, 1959, p. S118

[3.17] V.H. Rumsey,"Reaction Concept in Electromagnetic Theory ", Phys. Rev. 1954, pp. 1483-1491.

[3.18] V.H. Rumsey,"A Short Way of Solving Advanced Problems in Electromagnetic Fields and other Linear Systems", IEEE Trans. AP, 1963, pp. 73-86.

[3.19] David S. Saxon,"Tensor Scattering Matrix for the Electromagnetic Field" Phys. Rev., Dec. 1955, pp. 1771-1775.

[3.20] A.F. Stevenson, "Relation between the Transmitting and Receiving Properties of Antennas ",Quarterly of Applied Math., Vol. 5, #4, Jan. 1948 pp. 369-384

[3.21] W.J. Welch,"Reciprocity Theorems for Electromagnetic Fields Whose Time Dependence is Arbitrary"IRE Trans. AP, 1960, pp. 68-73.

IV. Point Symmetry Groups (Rotation and Reflection)

[4.1] C.E. Baum, "Interaction of Electromagnetic Fields with an Object which has an Electromagnetic Symmetry Plane" Interaction Note 63, March 1971.

[4.2] C.E. Baum, "A Priori Application of Results of Electromagnetic Theory to the Analysis of Electromagnetic Interaction Data", Interaction Note 444, 1985 and Radio Science, 1987, pp. 1127-1136.

[4.3] C.E. Baum,"Scattering, Reciprority, Symmetry, EEM, and SEM" Interaction Note 475, 1989.

[4.4] C.E. Baum, "SEM Backscattering", Interaction Note 476, 1989.

[4.5] C.E. Baum, "SEM and EEM Scattering Matrices, and Time-Domain Scatterer Polarization in the Scattering Residue Matrix", in "Direct and Inverse Methods in Radar Polarimetry", ed. W.M. Boerner, Reidel, in publication.

[4.6] C.E. Baum,"Combining RF Sources Using C_N Symmetry ", Circuit and Electromagnetic System Design Note 3, 1989.

[4.7] D.V. Giri and C.E. Baum, "Airborne Platform for Measurement of Transient or Broadband CW Electromagnetic Fields". Sensor and Simulation Note 284, 1984, and Electromagnetics, 1989, pp. 69-84.

[4.8] D.V. Giri and C.E. Baum, "High Frequency Capacitors",Energy Storage and Dissipation Note 10,1990.

[4.9] J.B. Knorr,"Tensor character of Symmetrical Waveguide Junctions", IEEE Trans. MTT, pp. 414-415.

[4.10] H.N. Kritikos,"Rotational Symmetries of Electromagnetic Radiation Fields", IEEE Trans AP, pp. 377-382, 1983.

[4.11] H.N. Kritikos,"Group Symmetries of Antenna Arrays", in H.N. Kritikos and Jaggard (eds)."Recent Advances in Electromagnetic Theory ,Springer Verlag,1990.

[4.12] D.M. Kerns,"Analysis of Symmetrical Waveguide Junctions", J. Res NBS, Vol 46, No.4, 1951, pp. 267-282.

[4.13] P. R. McIsaac,"Symmetry-Induced Modal Characteristics of Uniform Waveguides"I:"Summary of Results", pp.421-429, II:"Theory ",pp. 429-433 IEEE Trans MTT, 1975 pp 421-433.

[4.14] C.G. Montgomery, R.H. Dicke, and E.M. Purcell , "Principles of Microwave Circuits (Chapter 12) McGraw Hill", 1948.

[4.15] J. Nitsch,C.E.Baum,and R.Sturm,"The Treatment of Commuting Nonuniform Tubes in Multiconductor-Transmission-Line Theory",Interaction Note 481,1990.

[4.16] L.S. Riggs, J.M. Lindsey, and T.H. Shumpert,"Commonalities of the SEM Parameters of Several Simple Thin Wire Geometries", Electromagnetics, pp. 231-247,1989.

V. Space Symmetry Groups (Translation, Rotation and Reflection)

[5.1] A. Hessel, M.H. Chen, R.C.M.Li, and A. Oliner,"Propagation in Periodically Loaded Waveguides with Higher Symmetries" Proc. IEEE, 1973, pp. 183-195.

[5.2] J.B. Knorr and P.R. McIsaac, "A Group Theoretic Investigation at the Single-Wire Helix", IEEE trans. MTT, 1971, pp 854-861.

[5.3] P.J. Repean and P.R. McIsaac, "Consequences of Symmetry in Periodic Structures", Proc. IEEE, 1971,pp. 33-43.

[5.4] S.Sensiper,"Electromagnetic Wave Propagation on Helical Structures" Proc. IRE, 1955, pp 149-161.

VI. Babinet's Principle, Self Complementarity

[6.1] C.E. Baum and B.K. Singaraju,"Generalization of Babinet's Principle in Terms of the Combined Field to Include Impedance Loaded Aperture Antennas and Scatterers", Interaction Note 217, 1974.

[6.2] C.E.Baum,"The Self-Complementary Rotation Group", Interaction Note 479,1990

[6.3] H.G. Booker, "Slot Aerials and their Relation to Complementary Wire Aerials (Babinet's Principle)" , J.IEE, p. III-A, 1946, pp. 620-626.

[6.5] Y.G. Chen, S. Lloyd, R. Crumley, C.E. Baum and D.V. Giri, "Surface-Current-Density Measurements via Apertures", Sensor and Simulation Note 290, 1985 and Conf. Record of the Workshop on Measurement of Electrical Quantities in Pulse Power System II, March 1986, pp. 49-53, IEEE Dielectrics and Electrical Insulation Society, IEEE No. 86 CH 2327-5.

[6.6] R.S. Elliott, "A Note on Booker's Extension of Babinet's Principle", IRE Proc., 1952, p. 729.

[6.7] T.B.A. Senior, "Some Extensions of Babinet's Principle in Electromagnetic Theory", IEEE Trans AP, 1977 pp. 417-420.

VII. Chiral Media

[7.1] S. Bassiri, N. Engheta and C.H. Papas "Dyadic Greens' Function and Dipole Radiation in Chiral Media", Alta Frequenza LZ-N.2 pp83-88 ,1986.

[7.2] J.A. Kong,"Theorems of Bianisotropic Media",Proc.IEEE 60,1036-1046(1972)

[7.3] N.Engeta and D.L. Jaggard "Electromagnetic Chirality and Its Applications ," IEEE A.P. Newsletter October 1988 ,pp1-12.

VIII. Self Similarity , Log Periodicity

[8.1] G.A. Deschamps, "Frequency -Independent Antennas",chap.18 in H. Jasik(ed.) Antenna Engineering Handbook", McGraw Hill,1961.

[8.2] J.D. Kraus,"Antennas",2nd ed.,McGraw Hill,1988.

[8.3] P.E. Mayes,"Frequency-Independent Antennas", chap.9 in Y.T. Lo and S.W.Lee "Antenna Handbook",Van Nostrand Reinhold, 1988.

[8.4] V.H. Rumsey,"Frequency Independent Antennas", Academic Press ,1966.

[8.5] D.L. Jaggard. "On Fractal Electrodynamics", chap.6, pp. 183-224, in H.N. Kritikos and Jaggard (eds)."Recent Advances in Electromagnetic Theory , Springer Verlag,1990.

IX.General Texts and Monographs

[9.1] H.Bateman,"Electrical and Optical Wave Motion" Cambridge U. Press, 1955

[9.2] C.E. Baum and A.P. Stone "Transient Lens Synthesis;Differential Geometry in Electromagnetic Theory" Hemisphere Pubplishing Corp. 1990.

[9.3] L. Brillouin ,"Wave Propagation in Perodic Structures",Dover 1953.

[9.4] R.E Collin,"Foundations of Microwave Engineering", McGraw Hill,1966.

[9.5] J. Corwell, " Group Theory in Physics", Academic Press 1990.

[9.6] A.P Cracknell ,"Applied Group Theory" Pergamon Press ,1968.

[9.7] N. Gelfand et al., "Representations of the Rotation and Lorentz Groups and Their Applications". New York: MacMilan. 1963.

[9.8] M. Hamermesh,"Group Theory and Its Application to Physical Problems" Addison-Wesley, 1962.

[9.9] J.D. Jackson ,"Classical Electromagnetics",2nd ed. Wiley, 1975.

[9.10] J.A. Kong,"Theory of Electromagnetic Waves",Wiley Interscience,New York, 1975.
[9.11] G.A. Korn and T.M. Korn, "Mathematical Handbook for Scientists and Engineers", McGraw Hill,1968.
[9.12] R. Knox and A. Gold, "Symmetry in the Solid State" W.A. Benjamin Inc., 1964.
[9.13] L.D. Landau and E.M. Lifshitz " The Classical Theory of Fields", Addison Welsley,1971.
[9.14] G. Lyubarskii, "The Application of Group Theory in Physics". Elmsford, NY: Pergamon, 1960.
[9.15] C.W. Misner,K.S. Thorne, and J.A. Wheeler "Gravitation", W.H.Freeman and Company, San Franchisco ,1973.
[9.16] N.A. Naimark, "Linear Representation of the Lorentz Group", Elmsford, NY: Pergamon, 1964.
[9.17] N. Nussbaum, "Applied Group Theory for Chemists ,Physicists, and Engineers ", Prentice Hall 1971
[9.18] E.J.Post,"Formal Structure of Electromagnetics"North Holland,Amsterdam,1962.
[9.19] R.D. Richtmyer,"Principles of Advanced Mathematical Physics", Vol.II Springer Verlag, New York ,1981.
[9.20] A.V. Shubnikov and V.A. Koptsik,"Symmetry in Science and Art", Plenum Press 1974.
[9.21] M. Tinkham, " Group Theory and Quantum Mechanics", Mc Graw Hill ,1964.
[9.22] J. Van Bladel, "Relativity and Engineering", Springer Verlag,1984.
[9.23] H. Weyl, "Symmetry", Princeton University Press ,1952.
[9.24] P.B. Yale, "Geometry and Symmetry", Dover,1988.