Physics Notes

Note 11

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Application of Concepts of Advanced Mathematics and Physics to the Maxwell Equations

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Abstract

In order to advance electromagnetic theory, one can adopt techniques used in other fields, namely modern mathematics and physics, with suitable modifications. This gives new techniques for electromagnetic analysis, and especially for synthesis of new electromagnetic devices. This paper summarizes progress in this regard under five general headings: integral-operator diagonalization, complex variables applied to frequency, symmetry and group theory, differential geometry for transient lens synthesis, and electromagnetic topology.

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1. Introduction

Since the pioneering work of James Clerk Maxwell [17] in establishing what we call the Maxwell equations

$$
\begin{aligned}
\nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} - \vec{J}_h^{(s)} \\
\nabla \times \vec{H} &= \frac{\partial \vec{D}}{\partial t} + \vec{J} + \vec{J}^{(s)}
\end{aligned}
\tag{1.1}
$$

including both electric and equivalent-magnetic source terms, these have had a profound effect on the development of science and engineering. (Note that the divergence equations are implied by the curl equations.) In addition, some material-related parameters are needed to relate $\vec{J}$, $\vec{D}$ and $\vec{B}$ to $\vec{E}$ and $\vec{H}$, such as the constitutive parameters, for example in the form

$$
\tilde{\vec{J}} = \tilde{\overleftrightarrow{\sigma}} \cdot \tilde{\vec{E}}\,, \quad \tilde{\vec{E}} = \tilde{\overleftrightarrow{\varepsilon}} \cdot \tilde{\vec{E}}\,, \quad \tilde{\vec{B}} = \tilde{\overleftrightarrow{\mu}} \cdot \tilde{\vec{H}}
$$

$$
\sim \; \equiv \text{ two-sided Laplace transform over time t to complex frequency } s = \Omega + j\omega \tag{1.2}
$$

Here we have introduced the common frequency-domain form so that the vector fields are dot multiplied by 3 x 3 dyadic constitutive parameters, which in time domain become convolution operators over time. More general (even nonlinear) forms are sometimes encountered. Various boundary conditions (e.g., perfectly conducting surfaces) are readily derived as limiting cases.

People often think of dividing the basic and applied sides of the technological enterprise as between science and engineering, but this can lead to some confusion. I think that there is a better three-part division, which can shed some light on where electromagnetic (EM) theory fits into the structure. First, there is the basic scientific side which has electromagnetics as part of physics, and the fundamental question concerns the replacement of the Maxwell equations by something more accurate, applying to extreme conditions not normally encountered. This is not what we think of as electromagnetic theory in the usual sense. Second, we have what may be called applied science or basic engineering in which we explore the established physical laws (the Maxwell equations in this case) to see what they imply in the sense of discovering what is possible to analyze, synthesize, optimize, etc. This is distinct from the third category which might be termed applied engineering which concerns itself with the routine implementation of what is known from the second category in terms of technological products ("practicing" engineering), for example, by selection of antenna designs from a product catalog. Of course, the reader might prefer some other "diagonalization" but this should suffice for the present.

So, concentrating on the second category, the role of the electromagnetic theorist (including sometimes basic experiments, particularly as demonstrations and confirmations) concerns understanding what the Maxwell equations allow one to do in the way of analysis and synthesis of the performance characteristics of various electromagnetic devices as well as understanding the behavior of electromagnetic fields in natural environments. At this point, I would like to emphasize the concept of EM synthesis. One can analyze the interaction of EM waves with arbitrary geometries of various materials. While this is a challenging task, it is not synthesis. Synthesis starts with some desired performance characteristics and asks: "Is this possible within certain general constants (e.g., passivity)?" If it is possible, then one moves on to other questions such as : "What are the best possible values of the appropriate performance parameters?", and "What are the algorithms for designing (realizing) the device (antenna, scatterer, etc.) with the desired performance parameters?" By analogy one can recall that circuit analysis with passive lumped elements (LRC) was developed into a matrix form based on the Kirchoff laws for voltage and current as written on a network (graph). Circuit synthesis later asked (and answered) questions like [18]: "What kind of input impedances and transfer functions are possible in such networks", and "How are such things systematically realized?" An important part of EM theory then needs to be concerned with EM synthesis. One might even think of this as a generalization of circuit synthesis

In 1976 I published a review paper concerning transient EM theory [3]. In this I outlined some analytic concepts used in mathematics and physics that are not commonly being used, or just beginning to be used, in EM theory for both analysis and synthesis. Since then considerable progress has been made in exploring these concepts and obtaining useful results. In the present paper these analytical concepts and major results are summarized under the following section headings.

2. Integral-Operator Diagonalization

Electromagnetic scattering is often formulated as an integral equation of the form

$$\left\langle \overleftrightarrow{Z}(\vec{r},\vec{r}\,';s);\vec{J}(\vec{r}\,';s)\right\rangle = \vec{E}^{(inc)}(\vec{r},s) \tag{2.1}$$

The notation is related to bra/ket notation in quantum mechanics, with here integration over the common coordinates $(\vec{r}\,')$, type of multiplication (dot above the comma here), but with no conjugation implied since our operators are not in general Hermitian. For convenience (2.1) uses the symmetric impedance (or E-field) kernel, related to the dyadic Green function (of free space or other linear reciprocal media), but other kernels (e.g., H-field) are also used. The domain of integration can be over a volume or surface (using tangential components) as desired.

As with matrices for which one finds eigenvalues and eigenvectors we can form [7, 21]

$$\left\langle \overleftrightarrow{Z}(\vec{r},\vec{r}\,';s);\vec{j}_\beta(\vec{r}\,',s)\right\rangle = \tilde{Z}_\beta(s)\,\vec{j}_\beta(\vec{r},s) = \left\langle \vec{j}_\beta(\vec{r}\,';s);\overleftrightarrow{Z}(\vec{r}\,',\vec{r};s)\right\rangle$$

$\vec{j}_\beta(\vec{r},s) \equiv$ eigenmodes

$\tilde{Z}_\beta(s) \equiv$ eigenimpedances (eigenvalues)

$$\left\langle \vec{j}_{\beta_1}(\vec{r};s);\vec{j}_{\beta_2}(\vec{r},s)\right\rangle = 1_{\beta_1,\beta_2} = \begin{cases}1 \text{ for } \beta_1=\beta_2\\ 0 \text{ for } \beta_1\neq\beta_2\end{cases} \tag{2.2}$$

(orthonormal)

and we can refer to this as the eigenmode expansion method (EEM). For cases of degeneragy (two or more equal eigenvalues) one use the Gram-Schmidt orthogonalization procedure to complete the construction of the orthonormal set. (More on this appears under symmetry.) With (2.2) we can write the kernel in the form

$$\overleftrightarrow{Z}^{\nu}(\vec{r},\vec{r}\,';s) = \sum_\beta \tilde{Z}^{\nu}_\beta(s)\,\vec{j}_\beta(\vec{r},s)\,\vec{j}_\beta(\vec{r}\,',s) \tag{2.3}$$

where ν represents an arbitrary power, including $\nu=-1$ for inverse kernel which in (2.1) solves for the current on the scatterer. This is not the only kind of eigenmodes one can form from the integral equation, but is a natural choice for our purposes. Other kinds with other names (such as characteristic modes) are introduced by others for special purposes.

At this point we can recall [3, 7] that having solved for the eigenimpedances and eigenmodes of a perfectly conducting body (for which (2.1) becomes a surface integral equation), one can also solve directly for the body loaded by some uniform, isotropic sheet impedance $\tilde{Z}_\ell(s)$ by the transformation

$$\tilde{Z}_\beta(s) \rightarrow \tilde{Z}_\beta(s) + \tilde{Z}_\ell(s) \tag{2.4}$$

while retaining the same eigenmodes. Then $\tilde{Z}_\ell(s)$ can be synthesized to give desirable characteristics to the scatterer or antenna described in the form (2.1). Given $\tilde{Z}_\beta(s)$ for the unloaded body, then within the limitations of circuit synthesis one can make $\tilde{Z}_\beta(s) + \tilde{Z}_\ell(s)$ have desirable characteristics such as roots (poles of the response in which $\left[\tilde{Z}_\beta(s) + \tilde{Z}_\ell(s)\right]^{-1}$ appears) at desirable places in the s plane. These roots can even be made second order in some cases to give critical damping to the response. Here is a clear example of EM synthesis.

Here we also note that the $\tilde{Z}_\beta(s)$ can be split into interior and exterior parts (in electrical parallel combination) which separate the internal and exterior resonances (poles) [7]. However, the details are too elaborate to repeat here.

Recently [9] a transformation like (2.4) has been found to apply to more general volumetric dielectric bodies, even those consisting of homogeneous isotropic dielectric bodies residing in an inhomogeneous dielectric space.

3. Complex Variables Applied to Frequency

As discussed in [3] the analytic properties of the solution of the Maxwell equations as a function of the complex frequency, s, lead to several important ways to solve the Maxwell equations. For antennas and scatterers of finite size in three dimensions this leads to three methods based on expansions used in complex-variable theory.

3.1 Low-frequency method (LFM)

In complex variables functions are often expanded in terms of a power (Taylor) series about some point where the function is analytic. In EM, this is done for scattering by expanding about $s = 0$. As one expects, the leading terms are related to the induced electric and magnetic dipoles, related to the incident fields by polarizability dyadics. This is extended to antenna input impedance/admittance by inclusion of a pole at $s = 0$ when appropriate giving leading terms which can be interpreted as inductance, capacitance, and/or resistance [21].

Here, I would like to emphasize an application of importance to antenna design, particularly the low-frequency characteristics, concerns the matching of the electric- and magnetic-dipole moments ($\vec{p}$ and $\vec{m}$) in transmission [1]. Defining appropriate unit vectors we have

$$\begin{gathered}\vec{p}(t) = p(t)\,\vec{1}_p \quad , \quad \vec{m}(t) = m(t)\,\vec{1}_m \\ \vec{1}_p \cdot \vec{1}_m = 0 \quad , \quad m(t) = c\,p(t) \quad , \quad c = [\mu_o\varepsilon_o]^{-\frac{1}{2}} \\ \vec{1}_i = \vec{1}_p \times \vec{1}_m = \text{principal radiation direction (center of beam)}\end{gathered} \tag{3.1}$$

A remarkable property of such combined dipoles is that on the axis from the antenna "center" in the $+\vec{1}_i$ direction (beam center) the electric and magnetic fields are at right angles and related by

$$\frac{E}{H} = Z_0 = \left[\frac{\mu_0}{\varepsilon_0}\right]^{\frac{1}{2}} \text{ (wave impedance of free space)} \tag{3.2}$$

even in the near field including r^{-1}, r^{-2}, and r^{-3} terms, i.e., all the *dipole* terms. This has important consequences for low-frequency illumination of large areas for EM interaction measurements, such as for simulation of the nuclear electromagnetic pulse (EMP).

The pattern of such an antenna (far field) is a cardiod (radiated power proportional to $[1+\cos(\theta)]^2$ where θ is the angle to the observer relative to $\vec{1}_i$). It has a null in the back direction $(-\vec{1}_i)$, but there is a remaining r^{-3} term there with the same field ratio as in (3.2) (at right angles with Poynting vector still in the $+\vec{1}_i$ direction, i.e., back to the antenna). In reception such an antenna also has similarly interesting directional properties.

An important class of low-dispersion antennas (for transient/broad-band radiation/reception) are referenced as impulse radiating antenna (IRAs) [27]. These can be (and are) designed to exhibit this combined dipole behavior at low frequencies with $\vec{1}_i$ pointing in the same direction as the high-frequency beam. This improves the directionality and modestly decreases the low-frequency roll-off frequency.

3.2 Singularity expansion method (SEM)

Of more recent vintage (1971) there is SEM. There is already an enormous literature on this subject. Here we mention two review papers with lots of references [14, 15]. In this case a related basic complex-variable expansion is the Laurent expansions in which an expansion is found for the neighborhood of a pole.

From (2.1) natural frequencies and modes are found via

$$\left\langle \tilde{\overleftrightarrow{Z}}(\vec{r}, \vec{r}\,'; s_\alpha); \vec{j}_\alpha(\vec{r}\,') \right\rangle = \vec{0}$$

$$s_\alpha \equiv \text{natural frequency} \tag{3.3}$$

$$\vec{j}_\alpha(\vec{r}) \equiv \text{natural mode}$$

Immediately we observe that natural frequencies and modes have nothing to do with the incident-wave parameters (direction of incidence, polarization) in a scattering problem. To better appreciate the above, imagine that one is performing a moment method (numerical) computation. The kernel (operator) is replaced by an N x N matrix from which we find the natural frequencies via

$$\det\left(\tilde{Z}_{n,m}(s_\alpha)\right) = 0 \tag{3.4}$$

with the natural modes subsequently numerically determined. At this point, we can compare (3.3) to (2.2) and observe that the s_α are roots (zeros) of the $\tilde{Z}_\beta(s)$, linking the α index as β, β' (β'th root of the βth eigenvalue).

Assuming an incident plane wave as

$$\vec{\tilde{E}}^{(inc)}(\vec{r},s) = E_0 \tilde{f}(s)\, \vec{1}_p\, e^{-\gamma \vec{1}_i \cdot \vec{r}}$$

$f(t) \equiv$ incident waveform

$\gamma = \frac{s}{c}$ (in free space)

$\vec{1}_p \equiv$ polarization, $\vec{1}_i \equiv$ direction of incidence (3.5)

$\vec{1}_p \cdot \vec{1}_i = 0$

the current on the body is expanded as

$$\vec{\tilde{J}}(\vec{r},s) = E_0\, \tilde{f}(s) \sum_\alpha \eta_\alpha(\vec{1}_i, \vec{1}_p)\, \vec{j}_\alpha(\vec{r})[s-s_\alpha]^{-1} e^{-[s-s_\alpha]t_0}$$

+ possible entire function

$$\eta_\alpha(\vec{1}_i, \vec{1}_p) = U_\alpha\; \vec{1}_p \cdot \left\langle e^{-\gamma_\alpha \vec{1}_i \cdot \vec{r}\,'}, \vec{j}_\alpha(\vec{r}\,') \right\rangle$$

$\equiv$ coupling coefficient (3.6)

$$U_\alpha = \left\langle \vec{j}_\alpha(\vec{r}); \frac{\partial}{\partial s} \overleftrightarrow{Z}(\vec{r}, \vec{r}\,'; s)\Big|_{s=s_\alpha}; \vec{j}_\alpha(\vec{r}\,') \right\rangle^{-1}$$

$$\gamma_\alpha \equiv \frac{s_\alpha}{c}$$

This is the simplest form of coupling coefficient termed class 1, and it contains the information concerning the incident field. The entire-function term is applicable to early times. By judicious choice of the turn-on time t_0, it has been shown that, for perfectly conducting bodies, *this* entire function can be made zero [6]. In time domain the current is

$$\vec{J}(\vec{r},t) = E_0 f(t) \circ \sum_\alpha \eta_\alpha(\vec{1}_i, \vec{1}_p)\, \vec{j}_\alpha(\vec{r}) e^{s_\alpha t} u(t-t_0) \tag{3.7}$$

+ possible entire function (temporal form)

$\circ \equiv$ convolution with respect to time t

so that the pole terms transform to give a simple time-domain form. While the entire function can be made to be zero the sum in (3.7) is not an *efficient* early-time representation.

The scattered far field takes the form

$$\vec{E}_f(\vec{r},s) = \frac{e^{-\gamma r}}{4\pi r}\tilde{\overleftrightarrow{\Lambda}}(\vec{1}_o, \vec{1}_i;s) \cdot \tilde{\vec{E}}^{(inc)}(\vec{r},s)$$

$\vec{1}_o \equiv$ scattering direction (to observer at $\vec{r}$)

$\overleftrightarrow{1}_o \equiv \overleftrightarrow{1} - \vec{1}_o \vec{1}_o \equiv$ transverse dyadic at observer

$r \equiv \left|\vec{r}\right|$

$$\tilde{\overleftrightarrow{\Lambda}}(\vec{1}_o, \vec{1}_i;s) = -s\mu_0 \left\langle \overleftrightarrow{1}_o e^{\gamma \vec{1}_o \cdot \vec{r}} ; \tilde{\overleftrightarrow{Z}}^{-1}(\vec{r}, \vec{r}\,';s); \overleftrightarrow{1}_i e^{-\gamma \vec{1}_i \cdot \vec{r}\,'} \right\rangle \tag{3.8}$$

$$= \tilde{\overleftrightarrow{\Lambda}}^T(-\vec{1}_i, -\vec{1}_o;s) \text{ (reciprocity)}$$

$\equiv$ scattering dyadic

Using (2.3) one can readily express the scattering dyadic in EEM form. Here we write the SEM form as

$$\tilde{\overleftrightarrow{\Lambda}}(\vec{1}_o, \vec{1}_i;s) = \sum_\alpha \frac{e^{-[s-s_\alpha]t_0}}{s-s_\alpha} \vec{c}_\alpha(-\vec{1}_o)\vec{c}_\alpha(-\vec{1}_i) \ + \text{ entire function}$$

$$\vec{c}_\alpha(\vec{1}_i) = w_\alpha \overleftrightarrow{1}_i \cdot \left\langle e^{-\gamma_\alpha \vec{1}_i \cdot \vec{r}} ; \vec{j}_\alpha(\vec{r}\,') \right\rangle$$

$\overleftrightarrow{1}_i \equiv \overleftrightarrow{1}_i - \vec{1}_i \vec{1}_i \equiv$ transverse incidence dyadic (3.9)

$$W_\alpha = w_\alpha^2 = -s_\alpha \mu_0 U_\alpha$$

In backscattering this takes the symmetric form

$$\tilde{\overleftrightarrow{\Lambda}}_b(\vec{1}_i,s) = \tilde{\overleftrightarrow{\Lambda}}(-\vec{1}_i, \vec{1}_i;s)$$

$$= \sum_\alpha \frac{e^{-[s-s_\alpha]t_0}}{s-s_\alpha} \vec{c}_\alpha(\vec{1}_i)\vec{c}_\alpha(\vec{1}_i) + \text{entire function}$$

$$= \tilde{\overleftrightarrow{\Lambda}}_b^T(\vec{1}_i, s) \tag{3.10}$$

In contradistinction to the current in (3.6), except in special cases, the entire-function contribution to the scattering dyadic cannot be made to go to zero by judicious choice of t_0. Noting that the entire function is an early-time contribution one can look at the late-time response for target-identification purposes.

We can summarize the major areas of SEM development:

1. description of EM response (especially transient) of various structures (currents) modeling electronic systems [19]
2. equivalent circuits for antennas and scatterers [21]
3. target identification (free space) [15]
4. identification of buried targets (mines, unexploded ordnance) [26]

There are also various numerical techniques to analyze data for the SEM parameters [15]. Consulting the references one can find a huge list of references. Perhaps other major areas of SEM application will emerge in the future.

3.3 High-frequency method (HFM)

In complex-variable theory one often deals with asymptotic expansions as the complex variable tends to infinity. In EM we can collectively refer to such techniques as the HFM [3]. This includes geometric, spectral, uniform, etc., theories of diffraction. An enormous literature exists here. While I have had occasion to consider such techniques, these have developed by many others, and I will not dwell on this.

4. Symmetry and Group Theory

Group theory has long been used in physics to study the quantum mechanical properties of elementary particles, atoms, molecules, and crystal lattices based on the symmetries of the quantum wave functions. One may consider [3] whether something similar would be useful for the analysis and design of antennas and scatterers. Lewis Carroll had the Hatter ask: "Why is a raven like a writing-desk?" One might ask a similar strange question: "Why is an airplane like a hydrogen molecule?" At least the second question has an answer. They are like in two ways. A first way concerns SEM (Section 3.2). The natural frequencies s_α (in general complex) are characteristic of the body (homogeneous problem), and are analogous to the energy levels (typically real (bound states), but also complex (radioactive decay)) of the quantum system. A second way concerns symmetry. Both objects contain a symmetry plane and the EM response (eigenmodes and natural modes) and the quantum wave functions naturally divide into two sets (symmetric and antisymmetric) with respect to the symmetry plane. In physics this property is often called parity.

While the quantum symmetries are properties found in nature, the EM symmetries are of two kind: those inherent in the Maxwell equations (duality, reciprocity, relativistic invariance), and geometrical symmetries built into objects by human beings (or aliens). The close connection between the symmetries in antennas and scatterers and the symmetries in the associated EM waves can be used to design antennas and scatterers and to identify radar targets.

The reader can consult [23] for a detailed treatment of this subject, concerning which much progress has been made in recent years. Here we take a group in the form of a 3 × 3 dyadic representation as

$$\begin{gathered}
G = \left\{ \overleftrightarrow{G}_\ell \,\middle|\, \ell = 1, 2, \ldots, \ell_0 \right\} \\
\ell_0 = \text{group order} \quad \text{(finite or infinite)} \\
\overleftrightarrow{G}_\ell^{-1} \in G \;, \quad \overleftrightarrow{1} \equiv \text{identity} \in G \\
\overleftrightarrow{G}_{\ell_1} \cdot \overleftrightarrow{G}_{\ell_2} \in G \quad \text{for all ordered pairs of elements}
\end{gathered} \tag{4.1}$$

For the point symmetry groups (rotations and reflections) these are real and orthogonal with

$$\overleftrightarrow{G}_\ell^{-1} = \overleftrightarrow{G}_\ell^{T}$$

$$\det(\overleftrightarrow{G}_\ell) = \begin{cases} +1 & \text{proper rotation (no reflections)} \\ -1 & \text{improper rotation (includes a reflection)} \end{cases}$$

$$\overset{\leftrightarrow}{G}_\ell^{\eta_\ell} = \overset{\leftrightarrow}{1} \text{ (smallest } \eta_\ell) , \qquad \frac{\ell_0}{\eta_\ell} = \text{positive integer (for finite } \ell_0)$$

$$\lambda_n^{\eta_\ell}(\overset{\leftrightarrow}{G}_\ell) = 1 , \quad \lambda \equiv \text{eigenvalue} = \eta_\ell \text{ root of } 1$$

$$\eta_\ell \equiv \text{period of } \overset{\leftrightarrow}{G} \tag{4.2}$$

$$\det(\overset{\leftrightarrow}{G}_\ell) = -1 \Rightarrow \eta_\ell \text{ even for improper rotation}$$

In some cases these can be taken as 2×2 dyadics (or even scalars) (e.g., C_N for N-fold rotation axis).

By a symmetric body we mean one that is invariant under transformation by each element of the group. Transforming the body by

$$\vec{r}^{(2)} = \overset{\leftrightarrow}{G}_\ell \cdot \vec{r}^{(1)} \tag{4.3}$$

we require that the body be unchanged after this transformation (applying to every element of the group of interest). For the body constitutive parameters (permeability, permittivity, conductivity) represented as $\overset{\leftrightarrow}{\chi}$ we require

$$\overset{\leftrightarrow}{\chi}\left(\vec{r}^{(2)}\right) = \overset{\leftrightarrow}{G}_\ell \cdot \overset{\leftrightarrow}{\chi}\left(\vec{r}^{(1)}\right) \cdot \overset{\leftrightarrow}{G}_\ell^T \tag{4.4}$$

More generally, we can include the symmetries in the Maxwell equations in the transformation. For example, duality (interchange of electric and magnetic fields) can be included with the body symmetry to allow the interchange of permeability and permittivity dyadics (appropriately normalized) upon transformation by the group elements (self-dual body).

The EM fields are also transformed as in (4.3) except for a minus sign in the case of the magnetic field when the transformation has a reflection (improper rotation). The eigenmodes (2.2) and natural modes (3.3) are also transformed by the $\overset{\leftrightarrow}{G}_\ell$ while keeping the eigenvalues and natural frequencies unchanged. This leads to the symmetry-induced condition of eigenvalue (and natural frequency) degeneracy since $\overset{\leftrightarrow}{G}_\ell \cdot \vec{\tilde{j}}_\beta(\vec{r},s)$ is also an eigenmode for the same eigenvalue. In general, however, the eigenmodes so generated are not linearly independent. The number of independent eigenmodes for the same eigenvalue $\tilde{Z}_\beta(s)$ is the degree of degeneracy. For example,

C_N symmetry for $N \geq 3$ gives a two-fold degeneracy for $m \geq 2$ in the $\cos(m\phi)$, $\sin(m\phi)$ expansion in cylindrical coordinates. Small deviations from such symmetry break the degeneracy by giving small differences to the eigenvalues and natural frequencies, thereby leading to perturbation formulae.

Some of the recent symmetry results include:

1. placement and orientation of EM sensors on an aircraft to minimize the influence of aircraft scattering on the measurement (reflection symmetry R)
2. high-frequency capacitors (dihedral symmetry D_N)
3. nondepolarizing axial backscatter (two-dimensional rotation symmetry C_N for $N \geq 3$, e.g., an N-bladed propeller).
4. generalized Babinet principle (for dyadic impedance sheets) and self-complementary structures (C_{N_c} symmetry)
5. vampire signature (zero backscatter cross polarization in h, v radar coordinates) for mine identification (continuous two-dimensional rotation/reflection symmetry $O_2 = C_{\infty a}$) [10]
6. separation of magnetic-polarizability dyadic $\overleftrightarrow{\tilde{M}}(s) = \tilde{M}_z(s) \vec{1}_z \vec{1}_z + \tilde{M}_t(s) \vec{1}_z$ into distinct longitudinal and transverse parts, for low-frequency magnetic singularity identification (diffusion dominated natural frequencies) of metallic targets (C_N symmetry for $N \geq 3$)
7. categorization of the scattering dyadic for the various point symmetries, including reciprocity and self-dual case [28].

Other types of symmetry, such as translation, also have important consequences. These include common waveguiding structures and helices, as well as periodic structures (discrete translation). Dilation symmetries (continuous as in conical structures, and discrete as in log-periodic structures and fractal structures) also give special electromagnetic behaviors.

5.Differential Geometry for Transient Lens Synthesis

In gravitational theory differential geometry is used as an integral part of general relativity. In that case, one deals with a four-dimensional space/time. One can also use differential geometry in three spatial dimensions. In this case we are looking for coordinate transformations which allow us to take a known solution of the Maxwell equations with desirable properties in a relatively simple medium, and by curving the coordinates have the same solution in a nonuniform and perhaps anisotropic medium. Bending the wave propagation in this manner gives a lens. We think of this as a transient lens because this works equally well for all frequencies (within the limits of the practical realization of such a medium). For the case of a TEM mode (dispersionless) propagating along two or more guiding conductors the conductors are also curved in the coordinate transformation and are thereby positioned as boundaries on or inside the lens medium.

The theory with many examples is discussed in [22]. We imagine some as yet unspecified (u_1, u_2, u_3) orthogonal curvilinear coordinate system with

$$
\begin{aligned}
h_n^2 &= \left[\frac{\partial x}{\partial u_n}\right]^2 + \left[\frac{\partial y}{\partial u_n}\right]^2 + \left[\frac{\partial z}{\partial u_n}\right]^2 \quad \text{(scale factors, } n = 1,\ 2,\ 3\text{)} \\
[d\ell]^2 &= \sum_{n=1}^{3} h_n^2 [du_n]^2 = [dx]^2 + [dy]^2 + [dz]^2 \quad \text{(line element)}
\end{aligned}
\tag{5.1}
$$

The Maxwell equations as in (1.1) are taken in time domain as homogeneous, i.e., without sources. For time domain we then require zero conductivity with frequency-independent $\overset{\leftrightarrow}{\mu}$ and $\overset{\leftrightarrow}{\varepsilon}$. These fields and constitutive parameters are referred to as *real* (indicating they can be measured), as contrasted to the *formal* fields and constitutive parameters. These are designated by superscript primes such that

$$
\begin{aligned}
&\nabla' \times \vec{E}' = -\frac{\partial \vec{B}'}{\partial t}, \quad \nabla' \times \vec{H}' = -\frac{\partial \vec{D}'}{\partial t} \\
&\vec{B}' = \overset{\leftrightarrow}{\mu}' \cdot \vec{H}', \quad \vec{D}' = \overset{\leftrightarrow}{\varepsilon}' \cdot \vec{E}' \\
&\nabla' \times \vec{X}' = \begin{vmatrix} \vec{1}_1 & \vec{1}_2 & \vec{1}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ X_1' & X_2' & X_3' \end{vmatrix} \quad \text{(determinant notation)} \\
&\nabla' \cdot \vec{Y}' = \frac{\partial Y_1'}{\partial u_1} + \frac{\partial Y_2'}{\partial u_2} + \frac{\partial Y_3'}{\partial u_3}
\end{aligned}
\tag{5.2}
$$

The formal parameters define a problem in the u_n coordinates taken *as though* these were Cartesian coordinates. In tensor language the X'_n are the covariant components of $\vec{X}$ (applying to $\vec{E}$ and $\vec{H}$), while the Y'_n are the contravariant components of $\vec{Y}$ (applying to $\vec{D}$ and $\vec{B}$).

The formal and real fields are related by

$$\begin{gathered}
\vec{E}' = (\alpha_{n,m}) \cdot \vec{E} \ , \quad \vec{H}' = (\alpha_{n,m}) \cdot \vec{H} \\
\vec{D}' = (\beta_{n,m}) \cdot \vec{D} \ , \quad \vec{B}' = (\beta_{n,m}) \cdot \vec{B} \\
(\alpha_{n,m}) = (1_{n,m} h_n) = \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix} \\
(\beta_{n,m}) = \left(1_{n,m} \frac{h_1 h_2 h_3}{h_n} \right) = \begin{pmatrix} h_2 h_3 & 0 & 0 \\ 0 & h_3 h_1 & 0 \\ 0 & 0 & h_1 h_2 \end{pmatrix}
\end{gathered} \tag{5.3}$$

The formal and real constitutive parameters are related by

$$\overleftrightarrow{\varepsilon}' = (\beta_{n,m}) \cdot \overrightarrow{\varepsilon} \cdot (\alpha_{n,m})^{-1} \ , \quad \overleftrightarrow{\mu}' = (\beta_{n,m}) \cdot \overleftrightarrow{\mu} \cdot (\alpha_{n,m})^{-1} \tag{5.4}$$

which for diagonal constitutive-parameter matrices reduce to

$$\begin{gathered}
\overleftrightarrow{\varepsilon}' = (\gamma_{n,m}) \cdot \overleftrightarrow{\varepsilon} \ , \quad \overleftrightarrow{\mu}' = (\gamma_{n,m}) \cdot \overleftrightarrow{\mu} \\
(\gamma_{n,m}) = (\beta_{n,m}) \cdot (\alpha_{n,m})^{-1} = \begin{pmatrix} \frac{h_2 h_3}{h_1} & 0 & 0 \\ 0 & \frac{h_3 h_1}{h_2} & 0 \\ 0 & 0 & \frac{h_1 h_2}{h_3} \end{pmatrix}
\end{gathered} \tag{5.5}$$

For cases considered to date then we have

$$\overleftrightarrow{\varepsilon} = \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix} , \quad \overleftrightarrow{\mu} = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}$$

$$\overleftrightarrow{\varepsilon}' = \begin{pmatrix} \varepsilon_1' & 0 & 0 \\ 0 & \varepsilon_2' & 0 \\ 0 & 0 & \varepsilon_3' \end{pmatrix} , \quad \overleftrightarrow{\mu}' = \begin{pmatrix} \mu_1' & 0 & 0 \\ 0 & \mu_2' & 0 \\ 0 & 0 & \mu_3' \end{pmatrix} \tag{5.6}$$

where the components are referred to the u_n coordinates.

The problem is then to take some known formal fields with formal constitutive parameters, and find what u_n coordinates exist in which we have real fields and constitutive parameters subject to constraints (realizability conditions) on the real constitutive parameters. For example, one might be considering TEM waves propagating in the u_3 direction, making u_3 and ε_3 irrelevant. But then one might also like the real medium to be isotropic so that $\mu_1 = \mu_2$ and $\varepsilon_1 = \varepsilon_2$. It has been shown in such a case that constant u_3 surfaces are planes or spheres, limiting the class of acceptable u_n coordinates. Within this class various acceptable coordinate systems, and hence transient lenses, have been found.

As summarized in [22] there are several classes of solutions of these equations:

1. all six components of $\vec{E}$ and $\vec{H}$ nonzero for inhomogeneous but isotropic $\varepsilon, \varepsilon', \mu$ and μ' (only two possible coordinate systems)
2. TEM waves propagating in the u_3 direction for inhomogeneous but isotropic $\varepsilon, \varepsilon', \mu$ and μ' (coordinate systems constrained by constant u_3 surfaces being planes or spheres, examples including converging, diverging, and bending lenses)
3. two-dimensional lenses for TEM waves (only one component each of $\vec{E}$ and $\vec{H}$ nonzero) based on conformal transformations (resulting in only one of ε and μ being inhomogeneous, but both isotropic)
4. lenses with $\mu = \mu_0$ but ε anisotropic and inhomogeneous.

Since the book several new examples have been developed. An important class of these involve $\mu = \mu_0$ but ε inhomogeneous and isotropic, making them relatively practical for construction. Of these, an important type of medium is a cylindrically inhomogeneous dielectric (CID) with the permittivity distributed as

$$\frac{\varepsilon}{\varepsilon_{ref}} = \left[\frac{\Psi_{ref}}{\Psi} \right]^2 \tag{5.7}$$

in a cylindrical (Ψ, ϕ, z) coordinate system. This admits as solutions:

5. TEM waves propagating in the ϕ direction (bending lens) with very general transmission-line cross sections (e.g., circular coax) [2].

6. Electromagnetic Topology for Analysis and Control of Electromagnetic Interaction with Complex Systems

A certain kind of topology, graph theory, is commonly used in electrical engineering to describe electrical networks. For circuit analysis such networks are described by nodes and branches, on which are written the Kirchoff equations which say that the sum of the currents leaving a node are zero and the sum of the voltage drops around a loop are zero.

Electromagnetic topology (Fig. 6.1) begins by recognizing that space can be divided into a set of volumes separated by boundary surfaces. For signals to propagate from one volume to another they must pass through one or more surfaces. Some of these surfaces (closed ones) can take on the role of an EM shield in the usual sense. These can be nested inside one another to form a hierarchical topology. There is a dual topology, the interaction sequence diagram, which is a graph (or network) in which the volumes are replaced by nodes (vertices) and the surfaces separating adjacent volumes by branches (edges), this also being indicated in Fig. 6.1. This is the subject of *qualitative* (or descriptive) EM topology, which can be used to organize the EM design of complex systems. This is contained (along with quantitative aspects) in [5, 20] which also contain numerous references.

Quantitative EM topology is based on the BLT1 equation [4] which was originally stated in a form appropriate to multiconductor-transmission-line (MTL) networks as

$$\left[\left(\left(1_{n,m}\right)_{u,\mathrm{v}}\right) - \left(\left(\tilde{S}_{n,m}(s)\right)_{u,\mathrm{v}}\right) \odot \left(\left(\tilde{\Gamma}_{n,m}(s)\right)_{u,\mathrm{v}}\right)\right] \odot \left(\left(\tilde{V}_{(n)}(s)\right)_{u}\right)$$
$$= \left(\left(\tilde{S}_{n,m}(s)\right)_{u,\mathrm{v}}\right) \odot \left(\left(\tilde{V}_{n}^{(s)}(s)\right)_{u}\right)$$

$\left(\left(\tilde{S}_{n,m}(s)\right)_{u,\mathrm{v}}\right) \equiv$ scattering supermatrix

$\left(\tilde{S}_{n,m}(s)\right)_{u,\mathrm{v}} \equiv$ scattering matrix from vth wave into *u*th wave (nonzero only for junctions with this connection) ($N_u \times N_\mathrm{v}$)

$\left(\left(\tilde{\Gamma}_{n,m}(s)\right)_{u,\mathrm{v}}\right) \equiv$ characteristic propagation supermatrix (or delay supermatrix)

$$\left(\tilde{\Gamma}_{n,m}(s)\right)_{u,\mathrm{v}} = \begin{cases} e^{-\left(\gamma_{n,m}\right)_u L_u} & \text{for } u = \mathrm{v} \quad (N_u \times N_u) \\ \left(0_{n,m}\right) & \text{for } u \neq \mathrm{v} \quad (N_u \times N_\mathrm{v}) \end{cases}$$

$\equiv$ characteristic propagation matrix for waves on individual tube (uniform MTL) of length L_u (or delay matrix)

$\left(\left(1_{n,m}\right)_{u,\mathrm{v}}\right) \equiv$ supermatrix identity

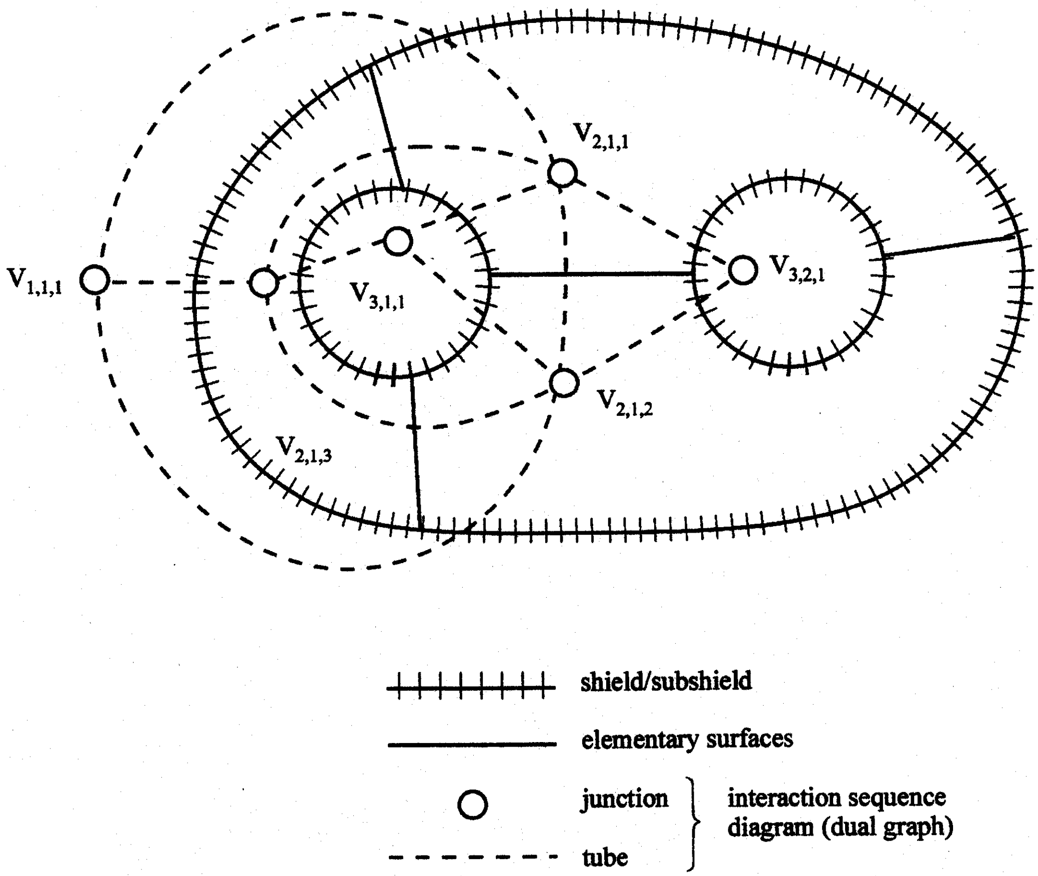

Fig. 6.1 Electromagnetic Topology (Hierarchical)

$$\left(\left(1_{n,m}\right)_{u,\mathrm{v}}\right) - \left(\left(\tilde{S}_{n,m}(s)\right)_{u,\mathrm{v}}\right) \odot \left(\left(\tilde{\Gamma}_{n,m}(s)\right)_{u,\mathrm{v}}\right) \equiv \text{interaction supermatrix}$$

$$\left(\left(\tilde{V}_n(s)\right)_u\right) \equiv \text{combined voltage supervector for waves leaving all the junctions}$$

$$\left(\tilde{V}_n(s)\right)_u \equiv \text{combined voltage vector } (N_u) \text{ at beginning of } u\text{th wave}$$

$$\left(\left(\tilde{V}_n^{(s)}(s)\right)_u\right) \equiv \text{combined voltage source supervector}$$

$$\left(\tilde{V}_n^{(s)}(s)\right)_u \equiv \text{combined voltage source vector } (N_u) \text{ for } u\text{th wave}$$

This is written on an MTL network (Fig. 6.2) consisting of junctions characterized by scattering matrices, and tubes consisting of MTLs with N_u conductors (plus reference) connecting appropriate junctions. Each tube contains two N_u vector waves, one propagating in each direction, indexed by w_u for $u = 1, 2, \ldots, N_w$ where N_w is twice the number of tubes.

The tubes here are taken as uniform (not varying along the tube) and characterized by

$$\begin{aligned}
\left(\tilde{Z}'_{n,m}(s)\right)_u &\equiv \text{per-unit-length impedance matrix for } u\text{th wave } (N_u \times N_u) \\
&= \left(\tilde{Z}'_{n,m}(s)\right)_u^T \qquad \text{(reciprocity)} \\
\left(\tilde{Y}'_{n,m}(s)\right)_u &\equiv \text{per-unit-length admittance matrix for } u\text{th wave } (N_u \times N_u) \\
&= \left(\tilde{Y}'_{n,m}(s)\right)_u^T \qquad \text{(reciprocity)}
\end{aligned} \tag{6.2}$$

From these we have

$$\begin{aligned}
\left(\tilde{\gamma}_{n,m}(s)\right)_u &= \left[\left(\tilde{Z}'_{n,m}(s)\right) \cdot \left(\tilde{Y}'_{n,m}(s)\right)\right]^{\frac{1}{2}} \text{ (positive real (p.r.) square root)} \\
&\equiv \text{propagation matrix for waves on individual tube} \\
\left(\tilde{Z}_{c_{n,m}}(s)\right)_u &= \left(\tilde{Y}_{c_{n,m}}(s)\right)_u^{-1} = \left(\tilde{\gamma}_{n,m}(s)\right)_u \cdot \left(\tilde{Y}_{n,m}(s)\right)^{-1} \\
&= \left(\tilde{\gamma}_{n,m}(s)\right)_u^{-1} \cdot \left(\tilde{Z}'_{n,m}(s)\right)_u \\
&\equiv \text{characteristic impedance matrix for } u\text{th wave}
\end{aligned} \tag{6.3}$$

In turn, the combined voltage waves are defined for each N_u wave by

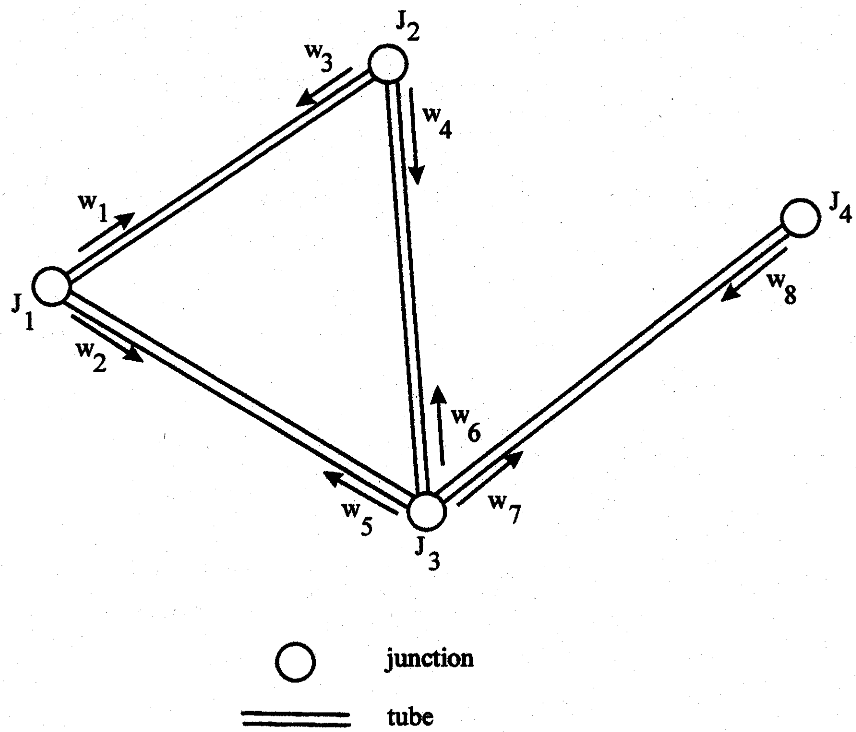

Fig. 6.2 Multiconductor-Transmission-Line Network

$$\left(V_n(z_u,s)\right)_u = \left(\tilde{V}_n(z_u,s)\right) + \left(\tilde{Z}_{c_{n,m}}(s)\right)_u \cdot \left(\tilde{I}_n(z_u,s)\right)$$
$$0 \le z_u \le L_u \tag{6.4}$$
$$\left(\tilde{V}_n(s)\right)_u \equiv \left(\tilde{V}_n(0,s)\right)_u$$

with positive convention for current in the direction of increasing z_u. (For the two waves on a tube (two values of u) the current conventions are *opposite*.) The distributed combined sources for the uth wave are similarly

$$\left(V_n^{(s)'}(z_u,s)\right)_u = \left(V_n^{(s)'}(z_u,s)\right) + \left(\tilde{Z}_{c_{n,m}}(s)\right)_u \cdot \left(\tilde{I}_n^{(s)'}(z_u,s)\right) \tag{6.5}$$

giving the source term in (6.1) as

$$\left(V_n^{(s)}(s)\right)_u = \int_0^{L_u} e^{-\left(\tilde{\gamma}_{n,m}(s)\right)_u [L_u - z_u]} \cdot \left(\tilde{V}_n^{(s)'}(z_u,s)\right)_u dz_u \tag{6.6}$$

Relating the MTL network to the EM topology, note that by shrinking the tubes to zero length the junctions can represent the volumes, the tubes the connecting surfaces, and the sources lumped equivalent sources at each surface. In this form (BLT2) then

$$\left(\left(\tilde{\Gamma}_{n,m}(s)\right)_{u,v}\right) = \left(\left(1_{n,m}\right)_{u,v}\right) \tag{6.7}$$

and disappears from (6.1). An alternate way to approach this is to recognize that a tube may be represented by a $2N_u$-port junction, fitting an MTL network into BLT2 form. A more elaborate form, the NBLT (nonuniform BLT) equation [8], allows for NMTL (nonuniform MTL) tubes for which the per-unit-length parameter matrices are allowed to vary as a function of z_u. In this last case the two N_u waves on a tube do not neatly separate, but scatter into each other as they propagate along the tube coordinate. Again this case can also be cast into BLT2 form by defining such a tube as a junction with scattering matrices and equivalent source vectors. A yet further form (BLT3) utilizes the delay property of the tubes to expand the interaction-supermatrix inverse in a geometric series which can be used for early-times in time domain [13].

These BLT networks can become rather elaborate for large electronic systems such as aircraft. Computer codes such as CRIPTE [24] have successfully modeled such systems, and further improvements are anticipated. The computation time has been recently significantly reduced by graph-theoretic techniques in which appropriate portions of the network are reduced to equivalent junctions before inverting the interaction supermatrix [11]. The successful implementation of such calculations has been from DC to several hundred MHz, pushing to a GHz. Further improvements may push this higher by modeling the cavities and cavities with transmission lines in appropriate ways that fit into the topologically-decomposed scattering-matrix formalism. Another potential improvement involves inclusion of the good-shielding approximation to break the full-system problem into smaller problems at shield/subshield boundaries, with simple matrix multiplication to reconnect the subproblems. One can also use SEM concepts to more simply evaluate the late-time behavior of the system in terms of natural frequencies and modes.

Closely tied to EM topology (although one could consider this a separate subject) is the subject of the response of NMTLs [25]. For this purpose it is convenient to formulate a single NMTL via a supermatrix differential equation of the form

$$\frac{d}{dz}\begin{pmatrix} \left(\tilde{V}_n(z,s)\right) \\ \left(\tilde{Z}_{n,m}(z,s)\right)\cdot\left(\tilde{I}_n(z,s)\right)\end{pmatrix} = \left(\left(\tilde{\xi}_{n,m}(z,s)\right)_{\sigma,\sigma'}\right)\odot\begin{pmatrix} \left(\tilde{V}_n(z,s)\right) \\ \left(\tilde{Z}_{n,m}(s)\right)\cdot\left(\tilde{I}_n(z,s)\right)\end{pmatrix} + \begin{pmatrix} \left(\tilde{V}_n^{(s)'}(z,s)\right) \\ \left(\tilde{Z}_{n,m}(s)\right)\cdot\left(\tilde{I}_n^{(s)'}(z,s)\right)\end{pmatrix}$$

$$\left(\tilde{Z}_{n,m}(s)\right) = \left(\tilde{Y}_{n,m}(s)\right)^{-1} \equiv \text{normalizing impedance matrix } (N\times N) \text{ chosen at our convenience} \tag{6.8}$$

$$\left(\left(\tilde{\xi}_{n,m}(z,s)\right)_{\omega,\sigma'}\right) = \begin{pmatrix} \left(0_{n,m}\right) & -\left(\tilde{Z}'_{n,m}(z,s)\right)\cdot\left(\tilde{Y}_{n,m}(z,s)\right) \\ -\left(\tilde{Z}_{n,m}(z,s)\right)\cdot\left(\tilde{Y}'_{n,m}(z,s)\right) & -\left(Z_{n,m}(z,s)\right)\cdot\frac{\partial}{\partial z}\left(\tilde{Y}_{n,m}(z,s)\right)\end{pmatrix}$$

where currents are referenced to the +z direction. Solving this equation gives relations between voltages and currents at both ends of the tube together with equivalent sources there. This is a chain-matrix-like formulation of the problem which is later (after solution) converted int a scattering-matrix form for insertion into the BLT equation. This equation is related to the supermatrizant differential equation

$$\begin{aligned}
\frac{\partial}{\partial z}\left(\left(\tilde{\Xi}_{n,m}(z,z_0;s)\right)_{\sigma,\sigma'}\right) &= \left(\left(\tilde{\xi}_{n,m}(z,s)\right)_{\sigma,\sigma'}\right)\cdot\left(\left(\tilde{\Xi}_{n,m}(z,z_0;s)\right)_{\sigma,\sigma'}\right) \\
\left(\left(\tilde{\Xi}_{n,m}(z_0,z_0;s)\right)_{\sigma,\sigma'}\right) &= \left(\left(1_{n,m}\right)_{\sigma,\sigma'}\right) \\
\left(\left(\tilde{\Xi}_{n,m}(z,z_0;s)\right)_{\sigma,\sigma'}\right)^{-1} &= \left(\left(\tilde{\Xi}_{n,m}(z_0,z;s)\right)_{\sigma,\sigma'}\right) \\
\left(\tilde{\Xi}_{n,m}(z,z_0;s)_{\sigma,\sigma'}\right) &= \left(\left(\tilde{\Xi}(z,z_1;s)\right)_{\sigma,\sigma'}\right)\odot\left(\left(\tilde{\Xi}(z_1,z_0;s)\right)_{\sigma,\sigma'}\right)
\end{aligned} \tag{6.9}$$

Provided that we have found the supermatrizant we have the solution of (6.8) as

$$\begin{pmatrix} (\tilde{V}_n(z,s)) \\ (\tilde{Z}_{n,m}(z,s)) \cdot (\tilde{I}_n(z,s)) \end{pmatrix} = \left((\tilde{\Xi}_{n,m}(z,z_0;s))_{\sigma,\sigma'} \right) \odot \begin{pmatrix} (\tilde{V}_n(z_0,s)) \\ (\tilde{Z}_{n,m}(z,s)) \cdot (\tilde{I}_n(z_0,s)) \end{pmatrix}$$

$$+ \int_{z_0}^{z} \left((\tilde{\Xi}_{n,m}(z',z_0;s))_{\sigma,\sigma'} \right) \odot \begin{pmatrix} (\tilde{V}_n^{(s)'}(z',s)) \\ (\tilde{Z}_{n,m}(z',s)) \cdot (\tilde{I}_n^{(s)'}(z',s)) \end{pmatrix} dz' \tag{6.10}$$

By choosing z_0 as one end of the tube and z as the other end the terminal parameters are related and the scattering supermatrix and equivalent sources are obtained.

The supermatrizant is expressed as a product integral [12]

$$\left((\tilde{\Xi}_{n,m}(z,z_0;s))_{\sigma,\sigma'} \right) = \prod_{z_0}^{z} e^{\left((\tilde{\xi}_{n,m}(z',s))_{\sigma,\sigma'} \right) dz'} \tag{6.11}$$

This can be thought of as a repeated dot product (increasing z' terms multiplying on left of form $e^{\left((\tilde{\xi}_{n,m}(z',s))_{\sigma,\sigma'} \right) \Delta z'}$), by comparison to the usual sum integral. In special cases, this reduces to a sum integral as

$$\left((\Xi_{n,m}(z,z_0;s))_{\sigma,\sigma'} \right) = e^{\int_{z_0}^{z} \left((\tilde{\xi}_{n,m}(z',s))_{\sigma,\sigma'} \right) dz'} \tag{6.12}$$

provided $\left((\tilde{\xi}_{n,m}(z',s))_{\sigma,\sigma'} \right)$ evaluated at z_1' and z_2' commute with each other for every pair z_1' and z_2' in the interval $z_0 \le z' \le z$. One example concerns a constant matrix, which gives the result for $(\tilde{\Gamma}_{n,m}(s))_{u,v}$ in (6.1). Another example concerns circulant matrices for the per-unit-length parameter matrices [16]. Special results also apply to the case of uniform modal speeds as occur for nonuniform wires (size, spacing) in a uniform medium [12, 25].

The product integral is suggestive of a numerical way for evaluating the supermarizant, i.e., by dividing the interval into some number of subintervals, approximating the result for each subinterval by assuming a constant matrix there, and multiplying the results for all the subintervals. This is a staircase approximation. One can do better in some cases by allowing a smooth variation (e.g., linear or exponential) of eigenvalues with constant eigenvectors over each subinterval [12, 25]. This allows one to preserve continuity of the line parameters from one subinterval to the nect, thereby reducing reflections at such boundaries.

The product integral has various special formulas analogous to those for sum integrals (e.g., integration by parts). What is called the sum rule allows one to separate $\left(\left(\tilde{\xi}_{n,m}(z',s)\right)_{\sigma,\sigma'}\right)$ into the sum of two terms. If one term has a readily evaluated product integral (closed form), the problem is changed to a new product integral. If the second term is suitably small, the new product integral can be readily approximated by the first two terms in a series representation of the matrizant (the first term being the identity) [25]. This gives a perturbation formula for approximating the solution of an *almost uniform* MTL.

7. Concluding Remarks

So we now have a collection of modern mathematical techniques to apply to the Maxwell equations (analysis and synthesis). Much has been learned using these and I would expect that much more can be learned. This should lead to new classes of electromagnetic devices.

We should continue searching for other mathematical structures which may be of use to electromagnetic theory. Noting the importance of the mathematics used in quantum mechanics, one might consider more esoteric physics such as quantum electrodynamics, string theory, etc. Not included in our discussion here, and still in its infancy is statistical electromagnetics, for which one may expect more important future results.

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